Preface

The most well-known methods for the exact analysis of boundary value problems for linear PDEs are the methods of (a) classical transforms, (b) images, and (c) Green's function representations. In spite of their tremendous range of applications, these methods have several limitations, for example: (A) For *second order* PDEs and separable boundary conditions, the Sturm–Liouville theory establishes the existence and also provides an algorithmic construction of an appropriate transform, but for higher order PDEs, or for nonseparable boundary conditions, there do *not* exist appropriate transforms. For example, there does not exist an appropriate x-transform for the Dirichlet problem on the half-line for an evolution PDE involving a third order derivative. (B) The method of images is restricted to those particular problems that admit certain symmetries. (C) The linear integral equations arising in the implementation of the method of Green's function representations are difficult to solve in closed form.

In addition to these obvious limitations, there exist additional, subtle, disadvantages. For example, although the Dirichlet problem of the heat equation on the half-line can be solved by the sine transform in x, the associated solution representation is *not* uniformly convergent at $x = 0$. Hence, it is not straightforward to verify that the solution satisfies the prescribed Dirichlet boundary condition, and furthermore, this representation does not provide an effective algorithm for the numerical evaluation of the solution. A similar difficulty exists for the sine-series representation of the Dirichlet problem in a finite interval. Furthermore, for the finite interval with two different Robin boundary conditions, the solution is expressed through a series which involves eigenvalues satisfying a transcendental equation.

The situation is even less satisfactory for boundary value problems with conditions of "changing type," for example Dirichlet in part of a boundary and Neumann in the remaining part. For such problems, since there does *not* exist an appropriate transform, one uses *some* transform such as the Fourier transform, and then one tries to formulate a so-called Wiener–Hopf problem.

A new method for analyzing initial-boundary value problems for *integrable nonlinear* evolution PDEs was introduced by the author in [1]. It was later realized that this method also yields novel integral representations for *linear* evolution PDEs. For example, it yields novel integral representations even for the classical problem of the heat equation on the half-line. The first implementation of the new method to linear PDEs was *not* presented in the simplest possible form. The first goal of this book is to provide a simple and self-contained presentation for the case of linear PDEs, with particular emphasis on the following four points:

1. Novel integral representations for the solution of initial-boundary value problems for evolution PDEs containing x-derivatives of arbitrary order and which are formulated either on the half-line or the finite interval are presented in Part I. For the case that the PDE involves third order derivatives, the only alternative to the method presented here is the use of the Laplace transform in t. The best way for the interested reader to appreciate the advantage of the new method is to attempt to solve, via the Laplace transform, an initial-boundary value problem on the half-line for a PDE in $u(x, t)$ involving u_t, u_x, and u_{xxx}. For evolution PDEs, the main advantage of the new method is that it constructs integral representations which (a) are uniformly convergent; and (b) involve integrals in the complex k-plane, which via contour deformation can be mapped to integrals containing integrands which *decay exponentially*. This construction, in addition to providing effective asymptotic results, also yields a novel numerical technique which appears to have several advantages over the standard numerical methods.

2. Novel integral representations for the solution of the Laplace, the Helmholtz, and the modified Helmholtz equations formulated in the interior of a convex polygon are presented in Part III. These representations provide the basis for the development of certain analytical and numerical techniques for solving these PDEs. The example of the Dirichlet problem for the modified Helmholtz equation in the interior of an equilateral triangle is discussed in detail; the solution is expressed in terms of an integral in the complex k-plane which involves an integrand which *decays exponentially*.

3. It is emphasized throughout this book that the new approach provides a unification as well as a significant extension of the classical transforms, of the method of images, of the Green's function representations, and of the Wiener–Hopf technique. Regarding the latter technique we note that through a series of ingenious steps, it finally gives rise to the formulation of a Wiener–Hopf factorization problem, which is actually equivalent to a Riemann–Hilbert (RH) problem. It is shown in Part IV that such RH problems can be *immediately* obtained using the *global relation*. This relation, which is an algebraic equation coupling certain transforms of all boundary values, plays a crucial role in the new method.

4. An interesting byproduct of the new approach is the emergence of an effective method for inverting certain integrals. This method, which is based on the formulation of either an RH or a d-bar problem, provides an alternative and simpler approach for deriving classical transforms (such as the Fourier, the Mellin, and the Kontorovich–Lebedev transforms). Furthermore, it has led to the inversion of the so-called attenuated Radon transform; this transform provides the mathematical basis of an imaging technique of major medical importance called single photon emission computerized tomography (SPECT); see Part II.

The second goal of this book is to show that for *integrable nonlinear* evolution PDEs, the new method yields novel integral representations formulated in the complex k-plane. These integrals, in addition to the exponentials which appear in the integrals of the linearized version of these nonlinear PDEs, also contain the entries of a 2×2 matrix-valued function $M(x, t, k)$, which is the solution of a matrix RH problem. The main advantage of this formulation is that the associated RH problem involves a jump matrix with *explicit exponential* (x, t) *dependence*, and thus it is possible to obtain effective asymptotic results using the Deift–Zhou (for the long-time asymptotics) and the Deift–Zhou–Venakides (for the zero-dispersion limit) techniques for the asymptotic analysis of these RH problems. The analysis of initial-value problems on the half-line for the nonlinear Schrödinger, the

Korteweg–de Vries, the modified Korteweg–de Vries, and the sine-Gordon equations, as well as the crucial role played by the associated global relations, are discussed in Part V.

The main results contained in this book are summarized in the introduction, which contains five sections, each of which summarizes the results obtained in the corresponding part of the book, i.e., section I.1 corresponds to Part I, etc. The introduction is rather long, but perhaps it provides an opportunity for the interested reader to assimilate quickly the essential results of the book, thus avoiding many computational details.

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Athanassios S. Fokas