Preface

Radar imaging is a subject of tremendous mathematical richness, with many interesting applications and many challenging open problems that are of a mathematical nature. Yet the topic is almost completely unknown in the mathematical community. This is because the field was developed within the engineering community, and the engineering literature is difficult for mathematicians to penetrate.

Mathematicians tend to like an axiomatic approach: they like to begin with one or more partial differential equations, say, the wave equation or Maxwell's equations, and derive everything from these equations. Physicists, on the other hand, are comfortable beginning with *solutions* of the fundamental equations. Engineers, in contrast, are able to begin with the key *part* of the solution, and proceed from there. Consequently mathematicians find it difficult to read the engineering literature because they don't understand where the *first* equation comes from. The goal of this monograph is to fill that gap and to show how radar imaging arises from the fundamental partial differential equations.

The focus here is on showing the connection between the physics and the mathematics, and on supplying an intuitive mathematical understanding of the basic ideas. Consequently, we ignore many issues of rigor, such as questions about the relevant function spaces and the precise conditions under which interchange of integrals is valid, because attention to these issues would only distract from the focus on the underlying ideas. We hope that this approach will provide a foundation that will enable mathematical readers to begin to read the extensive engineering literature and to start working in the field.

We have provided some references to ongoing work, but we have made no attempt at a complete literature survey or detailed history.

M.C. first developed this material for her Mathematical Sciences course "Introduction to Radar Imaging," a course that borrowed liberally from material prepared by B.B. for physics graduate students at the Naval Postgraduate School. M.C. then developed a ten-lecture version of the Radar Imaging course, first for a tutorial in September 2005 at the Institute for Mathematics and Its Applications, and then again for a lecture series in May 2008 at the University of Texas at Arlington. The latter lecture series was arranged by Tuncay Aktosun and was supported by the Conference Board of the Mathematical Sciences (CBMS). The present monograph corresponds to the CBMS ten-lecture series.

The assumed background for this material consists of the following.

- **A little complex analysis** (in particular Euler's relation $e^{i\theta} = \cos\theta + i\sin\theta$). We use a star ($*$) for complex conjugate.

- **The Fourier transform** [71, 97].

$$f(t) = \int e^{-2\pi i \nu t} F(\nu) d\nu = \frac{1}{2\pi} \int e^{-i\omega t} \tilde{F}(\omega) d\omega, \tag{1}$$

where $\tilde{F}(\omega) = F(2\pi\nu)$ with $\omega = 2\pi\nu$. There are different conventions regarding whether f is called the Fourier transform of F or $\tilde{F}$ or $\tilde{F}/(2\pi)$; these correspond to different choices about where to put the 2π, and we actually use different conventions at different places in the text. We use $\mathcal{F}$ to denote the Fourier transform operator.

  - We also need the inverse Fourier transform

$$F(\nu) = \int e^{2\pi i \nu t} f(t) dt \tag{2}$$

or, depending on the 2π convention,

$$\tilde{F}(\omega) = \int e^{i\omega t} f(t) dt. \tag{3}$$

There are also different conventions regarding whether it is the forward transform or inverse transform that has the minus sign in the exponent; the important thing is that the forward and inverse transforms have different signs.

  - A simple consequence of the definition is that when f is real-valued, its Fourier transform obeys the relation $F(-\nu) = F^*(\nu)$.
  - We also need the convolution theorem

$$(f * g)(t) = \int e^{-2\pi i \nu t} F(\nu) G(\nu) d\nu, \tag{4}$$

where the convolution is defined by

$$(f * g)(t) = \int f(t - t') g(t') dt'. \tag{5}$$

  - We also need the fact that smoothness of a function corresponds to rapid decay of its Fourier transform [71]; this connection can be seen by integrating the Fourier transform by parts.
  - The connection between smoothness and decay is closely related to the *Paley–Wiener theorem* [71], which states that a function $F(\omega)$ that is identically zero for negative ω has a Fourier transform that is analytic in the upper half-plane.
  - Finally, we need the n-dimensional Fourier transform and its inverse:

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} F(\boldsymbol{\xi}) d\boldsymbol{\xi}, \qquad F(\boldsymbol{\xi}) = \int e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}} f(\boldsymbol{x}) d\boldsymbol{x}. \tag{6}$$

- **The Dirac delta function** δ [54], which has the property

$$\int \delta(t) f(t) dt = f(0), \tag{7}$$

and the Fourier transform of the delta function,

$$\delta(t) = \int e^{2\pi i \nu t}\, d\nu = \frac{1}{2\pi} \int e^{i\omega t}\, d\omega, \tag{8}$$

are used repeatedly throughout the text. If t is replaced by $t - t'$ in (8), the resulting identity can also be thought of as a shorthand for the Fourier transform followed by its inverse.

- **The one-dimensional wave equation**

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

and its traveling-wave solutions [62], [132].

- **Facts from vector calculus**
  - Spherical coordinates.
  - If the vector field $\boldsymbol{B}$ has zero divergence, then $\boldsymbol{B}$ can be written as the curl of a vector field.
  - If the curl of a vector field $\boldsymbol{E}$ is zero, then $\boldsymbol{E}$ can be written as the gradient of a scalar function.
  - A few vector identities, including

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} \tag{9}$$

and the "BAC-CAB" identity

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}). \tag{10}$$

  - The divergence theorem

$$\int_\Omega \nabla \cdot \boldsymbol{V}\, d\boldsymbol{x} = \int_{\partial\Omega} \hat{n} \cdot \boldsymbol{V}\, dS \tag{11}$$

and its less well-known analogue [62]

$$\int_\Omega \nabla \times \boldsymbol{V}\, d\boldsymbol{x} = \int_{\partial\Omega} \hat{n} \times \boldsymbol{V}\, dS, \tag{12}$$

where $\partial\Omega$ denotes the boundary of Ω.

- **A little functional analysis**, in particular the following:
  - The definition of L^2 as the vector space of square-integrable functions, together with its inner product $\langle f, g\rangle = \int f(\boldsymbol{x}) g^*(\boldsymbol{x}) d\boldsymbol{x}$.
  - The adjoint of an operator. We denote the adjoint by a dagger.

– The Cauchy–Schwarz inequality:

$$\left| \int h(t) f^*(t) dt \right| \leq \|h\|_2 \|f\|_2, \quad \text{where} \quad \|f\|_2^2 = \int |f(t)|^2 dt \tag{13}$$

with equality only when h is proportional to f.

- **A little linear algebra:** Properties of orthogonal matrices are needed in Chapter 7.
- **A little familiarity with random processes:** The expected value [75], [101] is needed in Chapter 4 .
- **A little physics**:
  – The notions of frequency (denoted by ν), angular frequency (ω), wave speed (c), wavelength (λ), and wave number (k). The relationships $\nu = c/\lambda$, $k = \omega/c$ (in free space), $\omega = 2\pi\nu$, and $k = 2\pi/\lambda$ will be used.
  – Current, voltage, and Ohm's law

$$V = IR, \tag{14}$$

where V is voltage, I is current, and R is resistance.

The notion of a Green's function [119, 132] will also be needed in the text; readers are not assumed to be familiar with this notion, but those who have some familiarity with the concept will feel more comfortable.

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