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Variational Methods for the Numerical Solution of Nonlinear Elliptic Problems. *By Roland Glowinski.* SIAM, Philadelphia, 2015. $79.00. xx+462 pp., softcover. ISBN 978-1-611973-77-8.

This book originates from the lectures of the author at an NSF/CBMS conference and presents a general overview of numerical methods for nonlinear elliptic equations with an in-depth analysis of some specific techniques.

It starts with a quick guide to the variational formulation of elliptic problems and their approximation. Indeed, Chapter 1 has about 70 pages containing all the background, e.g., the main results on Hilbert spaces, trace theorems, the Lax–Milgram theorem, Galerkin and finite element methods, and Newton's method. It includes proofs of the most important results. The classical examples of Neumann and Dirichlet problems are extensively discussed, as well as variational inequalities of the first and second kind.

In Chapter 2 there is a detailed study of two fundamental tools: the Newton and conjugate gradient algorithms, which are described in the abstract variational framework of Banach/Hilbert spaces. As an example, the conjugate gradient iteration is applied to the Stokes system, on the pressure unknown, and on some simple nonlinear problems.

Chapter 3 concerns operator-splitting methods. The natural model problem is a time-dependent (parabolic) equation, which gives back the elliptic problem at the stationary limit when $t \to \infty$. Among the examples, the main focus is on Navier–Stokes, on which various splitting schemes are analyzed and tested numerically.

Chapter 4 deals with augmented Lagrangian formulations and the way to solve them by means of the alternating direction method. There are three important examples in this chapter. The first is the problem of the elastic large deformation of an inextensible beam. Static and dynamic simulations are discussed. The second example is the two-dimensional elliptic Monge–Ampère equation. Finally, an eigenvalue problem for visco-plasticity is considered.

Least-squares methods are the topic of Chapter 5. The framework is again the one of nonlinear problems formulated in Hilbert spaces. The solution method is conjugate gradient, which is generalized to the nonlinear setting. Examples here also include parameter-dependent problems, for which an arc-length continuation technique is introduced.

In the remaining part of the book, the previous techniques are combined in order to solve new classes of equations. Chapter 6 is about obstacle problems, Chapter 7 deals with the Lane–Emdel equation, $-\Delta u = \lambda u^3$, Chapter 8 is devoted to the Eikonal equation and variants, and Chapter 9 again addresses the Monge–Ampère equation and variants.

A strong point in favor of the book is that everything is shown and developed in the elegant variational setting, which gives a unitary view of the various techniques and ideas. There are indeed many links among different parts of the theory presented. Reading it is also enjoyable due to the presence of many historical remarks, both on the classics (e.g., the conjugate gradient algorithm or the Navier–Stokes equations) and on more specific and technical issues.

The book is dense in theory but, at the same time, it is suitable for any graduate student, since all the problems, methods, and results are developed from the beginning. As already observed, the presence of Chapter 1 also makes the book fully approachable for readers who do not have a background in variational methods.

I strongly recommend the book to anyone who wants to work in the field of numerical methods for PDEs.

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