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A Primer on Radial Basis Functions with Applications to the Geosciences. *By Bengt Fornberg and Natasha Flyer.* SIAM, Philadelphia, 2015. $79.00. x+221 pp., softcover. ISBN 978-1-611974-02-7.

We have, today, reached a point where most scientists in the broad area of computational and applied mathematics have some awareness of radial basis function (RBF) approximation. The fraction of those scientists who have actually applied RBF methods themselves is steadily growing, but only a rather small group is fully abreast of the current state of the art in RBF methods for solving partial differential equations (PDEs).

RBF methods were applied to PDEs for the first time in 1990. At that time, the RBF approximations were global and of the form

$$\tilde{u}(\underline{x}) = \sum_{j=1}^{N} \lambda_j \phi(\|\underline{x} - \underline{x}_j\|),$$

where $\underline{x}_j \in \mathbb{R}^d$ are scattered node points, λ_j are the degrees of freedom, and $\phi(r)$ is a (conditionally) positive definite RBF. Results were encouraging, but solving PDEs with global approximations led to systems of equations with dense matrices. Therefore, the computational cost and memory requirements become too demanding when applying global RBF methods to large scale or high-dimensional problems. Computational cost can be reduced by introducing compactly supported RBFs or using fast evaluation methods.

Ten years later, in 2000, the first localized RBF approximation method was introduced. By constructing a local RBF approximation for each scattered node point $\underline{x}_i$, $i = 1, \ldots, N$, based on $n \ll N$ neighboring points,

$$\tilde{u}(\underline{x}_i) = \sum_{j=1}^{n} \lambda_j \phi(\|\underline{x} - \underline{x}_j^{(i)}\|),$$

stencil weights for operators applied to the approximation can be computed and used for assembling global differentiation matrices. The resulting scattered node RBF-generated finite difference method is commonly referred to as RBF-FD.

Another decade later, around 2010, work was initiated on developing partition of unity RBF methods for PDEs (RBF-PUM). In this approach, local RBF approximations over P overlapping patches are blended through compactly supported partition of unity weight functions $w_k(\underline{x})$, $k = 1, \ldots, P$, to form a global approximation:

$$\tilde{u}(\underline{x}) = \sum_{k=1}^{P} w_k(\underline{x}) \sum_{j=1}^{n_j} \lambda_j^{(k)} \phi(\|\underline{x} - \underline{x}_j^{(k)}\|).$$

With some small concessions to the structuring of nodes, the two latter methods exhibit most of the advantages of global RBF approximations, but at a significantly reduced cost.

Several excellent books have been dedicated to the theory and practice of RBF interpolation and approximation, mainly focusing on the global case, which is becoming well understood. However, the book by Fornberg and Flyer is the first that covers the development and current state of RBF-FD methods. This is precisely the right time for such a book, when research has come far enough that the methods are ready for broader application in computational sciences.

Two of the main advantages of RBF methods are that they are flexible and easy to implement and that many artificial numerical problems such as singularities introduced by special coordinate systems can be avoided. When the unnecessary difficulties are removed, the focus can instead be on the real difficulties of advanced applications. Even more important, though, is the message that is brought home time and again in this book: RBF methods consistently outperform classical numerical approaches for large scale geophysical flow simulations. These results cannot be ignored and it is time that RBF methods are given a place at the high table of numerics alongside finite difference and finite element methods.

The book is aimed at readers with a background in numerical methods. It is perfect for someone who wants to understand how RBF methods fit into the broader picture

of numerical solution methods for PDEs. It is easy to read and to the point. Someone who wants to apply RBF-FD methods, especially to problems in geoscience, would benefit greatly from this work. Relevant citations for further reading are provided throughout the book and illustrative examples accompany each new concept. This is not a classical textbook with exercises at the end of each chapter, but as the authors suggest, it would be appropriate, e.g., for a full semester course for first year graduate students.

The authors claim more than once that they are favoring heuristics over rigorous theoretical derivations. This is true, but the book does contain some rigorous proofs. I would instead put it that the authors avoid spurious theoretical results, while pursuing those results that have direct practical implications.

It generally helps understanding to see how things are connected. This book follows the route from finite differences to spectral methods and then shows how RBF methods provide generalizations of both of these classes. Global RBF approximations are naturally covered first before moving to RBF-FD methods. Some common misconceptions are sorted out along the way and stable evaluation methods that are necessary in the regime of nearly flat RBFs are reviewed. Then, a number of challenging applications in geoscience are described in quite some detail, including the special measures needed to address problem-specific difficulties in each case. Unless the node locations are given by the problem, scattered node sets, possibly with variable density, need to be generated. Different strategies are discussed in an appendix.

No book can contain everything known about a subject, but this volume is a welcome contribution to the RBF literature that helps to organize dispersed results in the emerging RBF-FD field.

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