Preface

This book is focused on a powerful numerical methodology for solving PDEs to high accuracy in any number of dimensions: Radial Basis Functions (RBFs). During the past decade, this method has been shown to apply to a wide range of PDEs, arising, for example, in fluid mechanics, wave motions, astro- and geosciences, mathematical biology, computational electromagnetics, etc. In the past few years, the approach has advanced from being mainly "just another method that can be made to work on small toy problems" to one that can compete highly successfully against the very best previous approaches on some large benchmark problems. So far, its greatest successes in this direction have arguably been in the geosciences. Our focus in this monograph will be on how, when, and why RBF-based approaches work, more by means of examples and heuristic explanations than by rigorous theoretical arguments. Instead of trying to tread a careful path between the opposite sins of excessively intuitive arguments and formal rigor, we will systematically choose the former.

The RBF approach is generally attributed to Rolland Hardy [143], who in 1971 proposed it for the purpose of interpolating scattered 2-D data. However, some of the key theorems underlying its numerical stability go back to the 1930s (Buchner [20], Schoenberg [241]). It was recognized by Kansa [162, 163] in 1990 that the ability of RBFs to provide accurate approximations for derivatives of functions known only at scattered data locations offered a novel opportunity for the numerical solution of PDEs.

In this monograph, we will extend the "RBFs-for-PDEs story" both backward and forward in time. Conceptually, there is a very logical progression that starts with finite difference (FD) and pseudospectral (PS) methods and then, via global RBFs, leads to RBF-generated FD (RBF-FD) methods. While RBFs by now are quite well established, the RBF-FD approach is still an emerging methodology. Although the progression FD $\Rightarrow$ PS $\Rightarrow$ RBF $\Rightarrow$ RBF-FD is not quite how it always was perceived while the developments occurred, each of the last three methodologies is in fact closely linked to the preceding one. This book starts with brief introductions to FD and PS methods—limited to the extent that is needed for providing the perspective that we wish to convey about RBF and RBF-FD methods when these are applied to the task of solving PDE problems, in particular, as these arise in the geosciences.

Finite difference methods: These were first proposed for solving PDEs in 1911 [221], and they have remained a dominant methodology ever since. Generally, they are easy to implement but are more restrictive than, for example, finite elements in terms of geometric flexibility.

Pseudospectral methods: For applications in very simple geometries (intervals in 1-D, rectangular or circular domains in 2-D, periodic boxes in 3-D, spherical shells, etc.) it was noted in the early 1970s that the order of accuracy of FD methods often can be increased indefinitely and that this sometimes can offer spectacular computational efficiencies. Another way to arrive at the same PS methods is via expansions in orthogonal functions, such as Fourier, Chebyshev, and spherical harmonics (SPH). These PS methods soon became prominent for solving PDEs in numerous areas, including fluid dynamics (such as direct numerical simulations of turbulent flows), weather forecasting, long time evolution of linear and nonlinear waves, and computational electromagnetics.

Radial basis functions: It transpires that all PS methods can be seen as highly specialized (and typically not optimal) cases of RBFs applied to PDEs. RBFs generalize PS methods away from their severe geometric limitations and their dependence on very regular node layouts (which for PS methods makes it complicated to carry out local refinements in critical solution areas). This can be done while preserving their spectral accuracy (beyond any power of the typical node spacing). Both the coding effort and the computational cost of RBFs are independent of how simple or complicated the geometry might be. In a recent large-scale 3-D geophysical flow application, an RBF-based code on a standard PC [291] competed very favorably against all previous methodologies, even when these were implemented on large supercomputer systems. In spite of these successes, computational cost and "scalability" to large computer systems remained lingering concerns.

Radial basis function-generated FD methods: This takes us almost back to FD methods—where the numerical journey of the present monograph started. It has recently been discovered that using RBFs to create generalized FD methods might offer the best opportunity yet for combining the strengths of all the previous approaches. In particular, RBF-FD methods can offer (i) numerical stability even when using explicit time stepping of purely convective problems on irregular node layouts, (ii) very high computational speed (since they only rely on local approximations, and they also give rise to sparse rather than to full matrix problems), (iii) accuracy levels approaching those of PS and global RBF methods, (iv) easy opportunities for local (adaptive) refinements, and (v) excellent opportunities for large-scale parallel computing (from GPU boards to supercomputers with vast numbers of processors).

Applications of RBF and RBF-FD methods in the geosciences: While FD and PS methods by now have long histories, the RBF approach has only in the past few years taken the crucial steps up to full-blown applications and then further from just showing feasibility to demonstrating actual cost advantages, in some cases over all previously available methods.

This book follows quite closely the lectures that were given by the present authors at the NSF-CBMS Regional Research Conference "Radial Basis Functions—Mathematical Developments and Applications," held June 20–24, 2011, at the University of Massachusetts, Dartmouth. The lecture notes have here been edited and expanded so that this book also can serve as a textbook for a semester-length graduate course on RBFs (and, in particular, on their application to PDEs). Although some background materials are included here on ODE solvers, basic finite differences, etc., it is nevertheless recommended that students first complete some more introductory course on numerical methods before proceeding to the present material.

Acknowledgments: This book project would not have been possible without the generous help of many organizations and individuals. The Regional Research Conference was supported by NSF under the grant DMS 1040883. NSF has also provided individual support to the authors when developing much of the present materials. Furthermore, NCAR is supported by the NSF. The conference was superbly organized by Saeja Kim, Sigal Gottlieb, Alfa Heryudono, and Cheng Wang. Several colleagues have assisted not only with helpful discussions but also by giving detailed comments to early versions of the present manuscript. For this, we want especially to thank Nick Trefethen and Grady Wright. We also owe great thanks to SIAM and, in particular, to Sara Murphy, Elizabeth Greenspan, and Gina Rinelli for making the publication process run very smoothly and pleasantly.