Preface

Computational methods to approximate the solution of differential equations play a crucial role in science, engineering, mathematics, and technology. Indeed, the key processes which govern the physical world—wave propagation, thermodynamics, fluid flow, solid deformation, gas dynamics, electricity and magnetism, quantum mechanics, general relativity, and many more—are described by differential equations, and we depend on numerical methods for the ability to simulate, explore, predict, and control systems involving these processes. The variety of differential equation problems that arise in these applications is vast, and much research has gone into developing numerical methods which can solve different problems accurately and efficiently. Mathematical analysis of these algorithms plays an essential role, furnishing rigorous validation to particular methods in clearly delineated circumstances, supplying quantitative error bounds, and enabling comparison among competing methods. In this book we will focus on finite element methods, a vast class of numerical methods for differential equations which is of wide applicability and great utility, and also, not coincidentally, one for which there is an extensive body of mathematical analysis.

The finite element method is a mature tool, in both practice and theory, in many areas of computational science. Nonetheless, the variety of partial differential equations (PDEs) which arise is vast, and there are still many important problems for which the known numerical approaches fail, and good numerical methods are yet to be devised. Consequently, research aimed at devising and analyzing new methods is flourishing. Traditionally, the key mathematical tools for the study of numerical PDEs, and especially of finite element methods, have come from functional analysis: Hilbert and Banach spaces, the Hilbert projection theorem, the Lax–Milgram lemma, the Bramble–Hilbert lemma, duality, Sobolev spaces, etc. The finite element exterior calculus (FEEC), presented in this book, also depends essentially on functional analysis, especially the theory of closed unbounded operators on Hilbert space. But FEEC's mathematical arsenal goes well beyond functional analysis, bringing in tools from geometry and topology to develop and analyze numerical methods for classes of PDEs resistant to more traditional approaches. Methods derived from FEEC are prime examples of structure-preserving numerical methods, in that they are designed to preserve key geometric, topological, and algebraic structures of the PDE at the discrete level. This turns out to be crucial to the development of successful finite element methods for a variety of problems for which standard methods fail. Specifically, FEEC focuses on PDEs which relate to complexes of differential operators acting on Hilbert function spaces and uses finite element spaces which form subcomplexes of these complexes, and which can be related to them via commuting projections.

While FEEC's antecedents go back decades, to the early days of the finite element method and even before, it first began to be defined as a distinct theory in my presentation to the International Congress of Mathematicians in 2002 [5] and was formalized in two long papers I coauthored with Richard Falk and Ragnar Winther in 2006 [11] and 2010 [13]. The first paper emphasized a particular complex of differential operators, namely, the de Rham complex of differential forms on a domain in $\mathbb{R}^3$ (or a Riemannian manifold). It was here that the name *finite element exterior calculus* first appeared, referring to the calculus of differential forms. In the 2010 paper, more emphasis was put on the abstract structure of a Hilbert complex, of which the L^2 de Rham complex is a special case, allowing FEEC to deal with other complexes that arise in other applications.

By June 2012 the basic outlines of FEEC theory were in place, and I was fortunate to be offered the opportunity to present an intensive short course on FEEC to an audience of nearly 70 faculty members, graduate students, and other researchers from around the world. The course was generously supported by the National Science Foundation and the Conference Board of the Mathematical Sciences as part of the NSF-CBMS conference series and expertly hosted at the Institute of Computational and Experimental Research in Mathematics (ICERM) at Brown University. This book grew out of that course. It shares with the course the goal of helping numerical analysts to master the fundamentals of FEEC, including the geometrical and functional analysis preliminaries, quickly and in one place. But the book has a broader audience in mind than the course, aiming to be accessible as well to mathematicians and students of mathematics from areas other than numerical analysis who are interested in understanding how techniques from geometry and topology come to play a role in numerical PDE. FEEC has been vigorously developing in the time since the course, and so the book contains much more material than was taught in the course, some of which was not even developed at that time.

The first portion of the book, Chapters 1–5, quickly develops the prerequisite material from homological algebra, algebraic topology, and functional analysis. These ingredients are combined in the basic structure of a Hilbert complex studied in Chapter 4. Remaining in this general abstract framework, the approximation of problems related to Hilbert complexes is developed in Chapter 5. The second portion of the book consists of Chapters 6 and 7, where we apply the general theory to the most canonical example of a Hilbert complex, the L^2 de Rham complex on a domain in $\mathbb{R}^n$. Finally, in the closing chapter we briefly survey some other examples and applications.

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