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Introduction to the Numerical Analysis of Incompressible Viscous Flows. *By William Layton*. SIAM, Philadelphia, 2008. $67.00. xx+213 pp., softcover. Computational Science and Engineering. Vol. 6. ISBN 978-0-898716-57-3.

The incompressible Navier–Stokes equations are $\partial\mathbf{u}/\partial t + (\mathbf{u}.\nabla)\mathbf{u} = -\nabla p + \nu\triangle\mathbf{u}$, where the velocity field is denoted by $\mathbf{u}$, pressure by p, and kinematic viscosity by ν. The velocity field must be divergence free: $\nabla.\mathbf{u} = 0$. Subject to appropriate boundary conditions, the Navier–Stokes equations provide an adequate description of the flow of fluids such as water and air over a wide range of speeds. Although simple to write down, the equations possess very complicated solutions making both mathematical theory and numerical solution challenging.

The book under review by William Layton offers a tour of some of the mathematics and numerical algorithms connected with the Navier–Stokes equations, beginning with mathematical foundations and ending with models of turbulence. The book weaves together concise descriptions of experiments with elegant mathematics and numerical algorithms to tell a story that is engaging from beginning to end. It succeeds brilliantly and for the most part painlessly in its stated purpose of allowing graduate students to progress "from zero to finite element CFD" in a single semester. The book is written in the voice of a friendly teacher, making it a pleasure to read.

To get an impression of the character of the book, we may look at the first of three parts of the book in a little detail. This part is titled "Mathematical Foundations." Here we learn that a mathematical description of the velocity field usually requires at least two function spaces, one to capture the finiteness of the total kinetic energy and the other to capture local fluctuations. The mixing properties of a dyed jet injected into a fluid at a range of speeds gives us a sense of just what that means. We are soon told about L^2 spaces, weak derivatives, Sobolev spaces, the trace theorem, and the Ladyzhenskaya inequalities. The subtleties of these topics will elude readers without a prior grounding in analysis. The topics are lightly sketched and some simple lemmas are proved to help the reader get a sense of what is going on.

The first part includes a rapid introduction to the finite element method. This rapid introduction brings out the choice of degrees of freedom for finite-dimensional subspaces, coercivity of the underlying operator, and interpolation error estimates, using concrete examples where useful. Depiction of finite element meshes for different flow regimes gives a visceral sense of adaptive mesh refinement while omitting details. A facility with vector identities is a necessity for deriving and manipulating the Navier–Stokes equations. Thus, the first part ends by giving an account of vectors and tensors.

The principal difficulty in solving the Navier–Stokes equations numerically is in handling pressure. The difficulty is related to the zero divergence condition on the velocity field, but has little to do with the nonlinear nature of the equations. Indeed, the same issues arise for Stokes flow, where the nonlinear inertial term is ignored, or even in steady Stokes flow. The steady Stokes flow problem is to solve $-\triangle\mathbf{u} + \nabla p = \mathbf{f}$ for a given body force $\mathbf{f}$, subject to boundary conditions and the zero divergence condition on the velocity field $\mathbf{u}$. An important result (proved by Ladyzhenskaya) is the continuous inf-sup condition. Suppose Q is the space of L^2 functions on the bounded domain Ω with $\int_\Omega q = 0$ for $q \in Q$. Suppose X is the Sobolev space of velocity fields (not necessarily of zero divergence) which are square integrable with square integrable derivatives and which vanish on $\partial\Omega$. The inf-sup condition asserts the existence of $\beta > 0$ such that for every $q \in Q$ there exists $\mathbf{v} \in X$ with

$$(q, \nabla.\mathbf{v}) \geq \beta\, ||\nabla\mathbf{v}||\, ||q||.$$

This condition is intimately related to the existence of solutions to the steady Stokes problem. A discrete version of the inf-sup condition, which is labeled $\mathrm{LBB^h}$ after Ladyzhenskaya, Babuska, and Brezzi, is central to the validity of mixed finite element formulations for Stokes flow, as explained beautifully in this book.

The close connection established between mathematical theory and numerical algorithms is one of the strengths of this book. For example, the approximation of time-dependent flows is preceded by a sketch of the Leray theory. Where proofs are illuminating and not overly technical, they are given in more or less detail. As examples, we mention Poincaré's inequality and the proof of existence of a unique steady solution of the Navier–Stokes equations for small data. In the second instance, the uniqueness proof is connected to contraction mapping and the Oseen problem.

The last chapter is about turbulence. Flow visualizations are often employed to show that turbulence merits its name. Although very useful, they have not been rigorously connected to the velocity field. Indeed, information about vorticity obtained from flow visualizations can be misleading (see footnote 12 on page 12 of [3]). A much better way to get a sense of turbulence is to look at measurements of pointwise velocity obtained using hot-wire anemometers. The velocity field fluctuates wildly and irregularly in a boundary layer with a free stream velocity of just 15 ft/s or 10.22 mph [1]. Such fluctuations spanning a wide range of scales make turbulence modeling a practical necessity. The velocity field is decomposed into means and fluctuations, and turbulence models attempt to calculate the effect of the fluctuations using only the means. This is an endeavor fraught with logical difficulties because "there is no guarantee that in a strongly nonlinear problem means can be successfully calculated by equations containing only means" [2]. Yet the inevitability of having to model turbulence becomes evident when one realizes that it takes 500 to 1000 grid points in a direction normal to the surface to completely resolve the boundary layer at a surface moving through air at a speed of only 10.22 mph (the boundary layer is approximately three inches thick). Direct numerical simulation of turbulence from the Navier–Stokes equations is not possible in many engineering situations. The discussion of turbulence in this book is limited to some calculations of Kolmogorov, which revolve around the rate of energy dissipation per unit mass, and to eddy viscosity models.

REFERENCES

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