

Preface

But we are all led and guided by the passion to perceive and to understand, . . .
L. Euler, from the preface to: *Considerations on Nautical Problems*.

The accurate, efficient, and reliable simulation of problems involving the flows of liquids and gasses is necessary for scientific and technological progress in many areas. In fact, decisions which affect our everyday lives are made daily based upon computational simulations which are often performed for flow problems far beyond those which can be reliably computed! Computational fluid dynamics (CFD) is also an area in which appetite for computational resources has always exceeded supply and will continue to do so for the foreseeable future.

CFD is one of the current and central scientific frontiers, and there are still important contributions which will be made by mathematicians in this area. However, this is an area not easily accessible to mathematics students. Before reading current papers in the area, students need to learn analysis, functional analysis, partial differential equations, numerical analysis of partial differential equations, continuum mechanics, mathematical fluid mechanics, and so on. On top of this, they are expected to develop some physical understanding and insight into the physics of fluids.

The first known mathematical fluid mechanics book is *Hydrostatics* by the great Archimedes. Throughout history, mathematicians have been key contributors to the development of the understanding of fluid motion. Yet, in the current training of mathematics students, a few basic courses are taken, after which they work exclusively with one professor. It is little wonder that with each generation, mathematical researchers become more specialized and narrow! This natural progression makes it more difficult for each succeeding generation to reach the real scientific frontier of an area like CFD. Progress in CFD requires communication between experts in numerical analysis, fluid dynamics, and large-scale computing with constant comparison against the behavior of real fluids in motion.

This book was written to help graduate students who feel they are up to the challenge of the beautiful and complex world of CFD. The purpose of this book is to allow graduate students to progress from essentially zero to finite element CFD and even include one advanced topic in the field (such as turbulence¹) in one academic term. Because this book has been written for graduate students, there is a lot of repetition in the presentation. The

¹Turbulence is the example included as Chapter 10. There are many other important applications which, depending on the interests of the instructor, can be presented instead. Chapters 1 through 9 are designed to give students the foundation for many of them.

focus of this book is on incompressible viscous flows. This is the case best understood mathematically (and for which central issues are still unresolved). There are many other fluid flow problems whose extra difficulties build upon those of incompressible viscous flows, such as viscoelasticity, plasmas, compressible flows, coating flows, flows of mixtures of fluids, and bubbly flows. The world of fluid motion is fantastically varied and complex. A good understanding of the interconnections among the physics, the mathematics, and the numerics of the incompressible case is valuable, possibly even essential, for progress in these more complex flows.

For this purpose, this book must pick a path through finite element CFD which is both mathematically cogent and physically lucid. The path this book takes is energy (in)equality. The energy equality for a viscous incompressible fluid is a mathematical estimate fairly easily derived from the system of partial differential equations. More than that, it is the direct link between the Navier–Stokes equations (NSE) and the fundamental physics of fluid motion, stating in precise mathematical terms that

kinetic energy at time t + total energy dissipated up to time t balances initial kinetic energy + total kinetic energy input up to time t .

This energy equality implies *stability*² of the velocity: various norms of the fluid velocity are bounded by other norms of the problem data. Working backward from this fundamental physical fact and mathematical theorem, this book presents the mechanics of the equations of motion, their mathematical architecture, and the necessary analytical background. Working forward from the energy estimate, the book traces the energy norm path through the stability of finite element methods (FEMs) and their error analysis. In the last chapter, the K41 theory of turbulence is presented as a simple outgrowth of the energy equality. At the end, readers will have an intuitive yet mathematically rigorous connection from beginning to end. This thread might be thin after only one term, but students are then prepared to read (better) books and articles on the specific topics of the chapters and integrate what they have read within the overall picture of finite element CFD.

There are many challenges facing students beginning this study. It is always preferable to have a full and complete background before entry. While this is possible for some students, most find their background less than ideal. (Researchers in the area find it necessary to keep learning, too.) Many students in our program are still learning real analysis when they begin this book.

Several choices had to be made for this book. The first was to focus the proofs in the book purposely on the *proofs* the students must really master at their *first* entry to finite element CFD. Thus, when a result is important but its *proof* is not central, the result is quoted and the proof referenced to other books. The central proofs are presented with redundancy and extra explanations that experts could find tedious. The second choice regards the assumed analysis background of the student. These notes have tried to minimize this as far as possible, consistent with correctness and relevance. Chapter 1 begins with the unavoidable chapter on mathematical preliminaries. One choice in teaching a course based on this book is to postpone the material in this chapter (or at least the more technical

²There are many different types of stability of fluid motion, reflecting the needs of the variety of important applications of the area. Stability here is used in the simplest sense: the solution is bounded by the data of the problem.

subsections 1.1 and 2.1) until it is used and then introduce it bit by bit. (This is what I do.) The third choice was to write a book for *students* entering the field, assuming as little background as possible. This forces many interesting and important topics, such as duality, a posteriori error estimation, and adaptivity, to be left for a second treatment or further reading. Many students in our program have read (an early version of) this book on their own with minimal help and have gone on to do interesting work on many of these other topics.

The hardest choice about topics concerns what to do about implementation. This most important topic has been omitted for several reasons. The first one is simply time: an introduction (providing the foundation for reading, contributing as a mathematician, and communicating with other specialists) should be covered in one term. Second, the technical details about programming the methods become interesting mainly after seeing that the methods work (and work well) so extensions are needed. To see that the methods work well, there are excellent and easy-to-use programs available for finite element CFD; for example, COMSOL Multiphysics (previously FEMLAB) and FREEFEM++ are very friendly to students. In going from two to three space dimensions, real difficulties arise which require parallel codes. The code ViTLES, developed by Traian Iliescu and Jeff Borggaard, is a parallel, three-dimensional, NSE platform for both laminar and turbulent flows. My friends Vince Ervin and John Burkardt (Flow7) both have elegant finite element CFD programs on their web pages that my students have benefitted from.³ The third reason for not presenting implementation at length herein is that implementation of the FEM for the NSE certainly deserves a book of its own!

I have taught a course based on this book with success at the University of Pittsburgh. This class included many beginning graduate students with interest (but not background) in analysis and applied mathematics and some more advanced graduate students already doing research in other areas of computational mathematics. For almost all of these students, this course was their first exposure to fluids, and for many of the students, the course was their first exposure to analysis beyond that of a typical beginning graduate student. In addition, I have had many students from engineering and physics who have enjoyed the mathematical presentation of CFD. All these students were users of CFD technology in their research and were led to seek a more systematic understanding of *why*, *when*, and *how* typical algorithms (do and don't) work. This book is far from appropriate for the first exposure of an engineering student (for example) to CFD but an excellent later course.

For success, I found three factors essential. The first is the commitment of students. They should understand that they will be learning a lot and that in true learning one is never in one's comfort zone. The second factor is active learning: students should try to work at least one exercise regularly after each section or lecture. (I include a few carefully chosen exercises in the text.) The third factor is a glimpse of the broader world of computational fluid dynamics beyond these notes. This can be done in many ways to fit the interests of the students and teachers. One way that is exciting for everyone is to have groups explore some of the available FEM CFD programs and report periodically.

Welcome to the exciting and beautiful world of fluids in motion and finite element computational fluid dynamics!

³www.freefem.org, www.icam.vt.edu/ViTLES, www.csit.fsu.edu/~burkardt/f_src/flow7/flow7.html

Overview of the Topics

Two fundamental mathematical arguments that provide support for finite element approximations of the NSE are

1. convergence of the approximate velocity and pressure of the steady NSE for laminar flows (i.e., at small enough Reynolds number to ensure global uniqueness) and
2. convergence in the energy norm of the semidiscrete approximation to the solution of the time-dependent problem.

Much research on the numerical analysis of the NSE is a reaction to the limitations of these two arguments or an elaboration of them to new problems and algorithms. These convergence results are themselves extensions of three (far more important) stability results:

1. The approximate pressure of the Stokes problem is bounded by problem data under the discrete inf-sup condition.
2. The approximate velocity of the steady problem is bounded by problem data if the nonlinearity is explicitly skew-symmetrized.
3. The kinetic energy in the approximate velocity of the time-dependent problem is bounded by problem data with the same treatment of the nonlinearity.

These three evolved from the corresponding energy estimates for the NSE which express a fundamental and direct link to the physics of fluid motion. Working backward to the preparation of a typical student entering graduate school in the mathematical sciences, focusing on what convergence analysis can tell about fluid flow phenomena (and what it leaves unsaid as well), the topics and order in this book emerged. The book's goal is to present a connected thread of ideas in the numerical analysis of the NSE without losing sight of understanding what fluid flow simulations really mean. Since the background of students is highly variable, each chapter is also presented to be as self-contained as possible so that students can begin at their appropriate place in the book. This means that essential definitions and results often reappear in chapters after they are first introduced.

Inevitably, this book begins with a chapter called Mathematical Preliminaries. Such a chapter must be there for a book to be read by a student without a teacher to supply gaps in the student's background. There are the usual choices: cover the chapter in detail, skip this chapter, totally introducing the topics in Chapter 1 as they are needed, or just introduce $L^2(\Omega)$ and $H_0^1(\Omega)$ and move ahead quickly, introducing the other topics and results as needed. (This is what I do.) Students are often anxious to see methods and applications before believing that theory is justified. Chapter 2 introduces (as quickly and as cleanly as possible) the FEM in enough detail to begin the treatment of essential ideas in the NSE. Many students (but not all) will have had some experience in the FEM. Chapter 2 is for students who are new to the FEM (and is not a substitute for a course on the topic). It presents energy norm stability and convergence of the FEM for the linear element and small extensions. The extensive, important, and intricate theory of the FEM is motivated by exactly this case, so this is the place to start and it is enough. Chapter 2 does not take the path (traditional since the finite element book of Strang and Fix) of beginning in one dimension

then repeating the presentation in two dimensions. This approach (which is best for a full term course on FEMs) takes a lot of time. I have found the key to students understanding the FEM at some intuitive level is pictures, drawings, schematics, etc., of meshes and basis functions, all possible for linear elements on triangles in two dimensions. Thus, the FEM chapter begins in two dimensions.

Engineering and science students are (in my experience) adept at calculations with vectors and tensors, but, alas, these topics are treated only very briefly in some undergraduate mathematics curriculums.⁴ Thus, Chapter 3 covers and reviews vectors, tensors, and conservation laws. It presents only that which is necessary to go farther in CFD.

Part II opens with Chapter 4, which presents mixed methods for the Stokes problem. The message of Chapter 4 is centrality of stability of the discrete pressure. This leads to the continuous inf-sup condition and its discrete analogue. Every chapter in this book has multiple excellent books on the chapter's topic, and Chapter 4 is no exception. By emphasizing stability of the discrete pressure, the treatment of mixed methods is shortened and much of the important and beautiful theory of mixed methods is left for the students' next steps in the field. Still, a clear understanding of

$$LBB^h \Rightarrow \text{stability of } p^h \Rightarrow \text{convergence}$$

is essential for reading and understanding the theory of mixed methods. It may seem odd to place the Stokes problem before the derivation of the equations of fluid motion. This was done to keep the students focused on numerics. Each theory chapter is followed by a numerical analysis chapter connected to and expanding the abstract theory. The numerical analysis of the Stokes problem can be presented before the derivation of the NSE, and to delay discussion of numerical issues longer risks losing the interest of many students.

Chapter 5 gives a derivation of the NSE and discusses the properties of solutions that are essential to understand for their numerical solution. Many topics are streamlined here, too, with the time constraints of one term in mind. For example, in the treatment of boundary layers, only the derivation of the $O(\text{Re}^{-\frac{1}{2}})$ estimate of the width of a laminar boundary layer is given. This estimate is important to understand for mesh generation and for estimating Re dependence of errors in a simulation. The boundary layer equations are omitted although they are so very close at hand after deriving this estimate.

Chapter 6 presents the essential theory of the steady NSE. The stability (meaning here that velocity is bounded by body force) of the velocity in the steady NSE is the connection between the mathematical architecture of the steady NSE and the physics of fluid motion. Uniqueness of the steady solution for small data is proved in Chapter 6 in a manner that introduces the steps in the convergence proof of Chapter 7 in a simplified setting.

The numerical analysis of the steady NSE is developed in Chapter 7 as a natural evolution of the stability bound for the velocity presented in Chapter 6. This convergence proof is essentially a simplification of the one in Girault and Raviart's wonderful 1976 monograph. The constants in the final result are not as sharp as those in the proof by Girault and Raviart because of these simplifications. For most students at this stage of their studies, this convergence proof will be the most complex one they will have struggled with. It is developed in steps in Chapter 7 and simplified to help students begin to see it as an

⁴One math student, who is now an accomplished applied analyst, told me that the part of the course that helped him the most was learning the summation convention!

elaboration of a few simple themes. In particular, it is presented as a variation on the proof of stability of the velocity and pressure of the continuous problem. Finally, the small data condition is interpreted for the time-dependent NSE.

Part III considers time-dependent fluid flow beginning with a summary of the Leray theory of the NSE in Chapter 8. It is my experience that proofs can be postponed, but any further simplification of the presentation of the theory results in much confusion that requires more time to correct later than is saved in the present. There are no useful shortcuts here: energy inequality, weak solution, strong solution, and uniqueness conditions are all needed, and all address essential physical issues relevant for computations. The Leray theory is built upon the physical foundation of the energy equality

$$\begin{aligned} \text{kinetic energy}(t) + \text{total energy dissipated over } [0, t] \\ = \text{kinetic energy}(0) + \text{total power input over } [0, t]. \end{aligned}$$

This is the most important and direct connection between fluid flow phenomena and the abstract Leray theory of the NSE. Chapter 8 proves stability of finite element methods for the time-dependent NSE by showing that the approximate solution satisfies the above energy equality. Convergence is studied in the energy norm. This is the most fundamental convergence analysis. It is a natural extension of the energy equality from the Leray theory of Chapter 8.

At this point, the topics could naturally end. One final topic is presented from among the enormous variety of fluid flow phenomena at the leading edge of CFD: turbulence. Some accepted physical theories of turbulence are easily accessible to students. These have not yet had impact in the numerical analysis community matching their physical importance and the insight they provide. Turbulence is also one of my own fascinations and research interests. I have restrained (with difficulty) the presentation of turbulence in Chapter 10 to homogeneous, isotropic turbulence and eddy viscosity models. Although not the leading edge, these are core ideas which still influence much current research. Indeed, much of the current research on numerical simulation of turbulent flows is aimed at attaining the good stability properties of (discretizations of) eddy viscosity models while avoiding their ad hoc nature, over damped effects on solutions and inaccurate predictions of many turbulent flows.

The field of fluid mechanics is wonderfully diverse. Fluids comprise three of the four states of matter, and the equations of fluid motion provide good models of many solids that flow as well (such as traffic and granular materials). There are also many materials that are important for industrial processing and manufacturing that sometimes behave like a solid and sometimes like a fluid! Laminar, isothermal, internal flows of a single, homogeneous, Newtonian liquid form only a fraction of fluid flows for which accurate predictions are needed. More complex flows add many layers of difficulties on top of those considered herein. Nevertheless, the understanding of incompressible, viscous flows is essential for progressing to more complex flows. There are also very many open questions and uncertainties in the numerical simulation of the case of laminar, isothermal, internal flows of a single, homogeneous, Newtonian liquid!

If nature were not beautiful, it would not be worth studying it. And life would not be worth living.

J. H. Poincaré, 1854–1912, quoted in G.W. Flake, *The Computational Beauty of Nature*, M.I.T. Press, 2000.

Acknowledgments

It is a pleasure to thank many people for their help in developing these notes. The reviewers of an early version provided detailed comments, far beyond normal expectations in a usual review. I appreciate their reviews, which were very helpful and definitely improved this book. My friends Vince Ervin, John Burkardt, and Noel Walkington gave me a lot of useful feedback (for which I am grateful) on various versions of this book and for some of the examples of FEM meshes (provided by Vince), the derivation of the energy equation in Example 3.3.6 (by Noel), and figures of finite element basis functions (by John). Other interesting and important figures were provided by Carolina Manica (the graph of energy dissipation rate versus kinetic energy), Leo Rebholz (some three-dimensional flow visualizations), Songul Kaya-Merdan (the driven cavity example), and Monika Neda (all the examples of flow around cylinders and the step flows simulations). I am grateful to Milan Jevtic, who helped by converting crude, hand-drawn schematics to professional-quality graphics. I appreciate all their help immensely. I thank my colleagues and friends at the University of Pittsburgh, Patrick Rabier, Paolo Galdi, Beatrice Riviere, Ivan Yotov, Chuck Hall, and Tom Porsching, for many vigorous and illuminating discussions on these topics. I owe a great debt to the graduate students upon whom the various versions of these notes were inflicted and a greater debt still to Max Gunzburger, who introduced me to the beautiful world of mathematical fluid dynamics years ago.