Preface

Inverse problems arise from the need to interpret indirect and incomplete measurements. As an area of contemporary mathematics, the field of inverse problems is strongly driven by applications and has been growing steadily in the past 30 years. This growth has been fostered both by advances in computation and by theoretical breakthroughs. Modern digital sensors provide vast amounts of data related to diverse areas including engineering, geophysics, medicine, biology, physics, chemistry, and finance. As a result, the need for computational inversion can be expected to increase in the future.

The main goal of this book is to provide a practical introduction to inverse problems from both a computational and theoretical perspective. A solid theoretical framework is mandatory for understanding why *ill-posed inverse problems* require a different set of solution methods than well-posed problems. Ill-posedness is related to interpretation tasks that are extremely sensitive to measurement and modeling errors. On the other hand, solving an inverse problem involves the implementation of a computational algorithm that recovers useful information from measured data (the word “useful” can best be understood in the context of a particular application). A successful inversion algorithm is robust against measurement noise, computationally effective, and mathematically justified by appropriate analysis and theorems.

Much of the literature on computational inversion considers tailored methods for linear problems (such as filtered back-projection for X-ray tomography) and generic iterative methods for nonlinear problems (such as Tikhonov regularization with nonlinear objective function). However, in this book we do exactly the opposite: we discuss a unified solution framework for linear problems and tailored direct methods for nonlinear problems. Our rationale is the following:

- Linear inverse problems are all essentially alike since they are completely described by the singular value expansion of the forward map. Thus it makes sense to apply a general methodology designed for complementing measurement data with a priori information, for example, by enforcing nonnegativity or by promoting sparsity in a basis.

- Nonlinear inverse problems are all different and need dedicated solution methods. One way to proceed is to use the results of the analytic-geometric inverse problems research tradition to construct regularized algorithms.

The book is organized into two parts. The first part, “Linear Inverse Problems,” is suitable for a one-semester undergraduate course or for a part of a graduate course. We present both continuous and discrete inverse problems to instruct how the ill-posedness is inherent in the idealized inverse problem and how it shows up in the real-life problem and in its discretization. With this approach we hope the reader will develop a deeper understanding of the connection between the mathematical theory, the computational model, and the practical problem arising from the application.

The guiding examples in Part I are the problem of image deblurring, X-ray tomography, and backward heat propagation. We discuss how to realistically simulate measurement data for all three. A dangerous pitfall in algorithm development and testing is the act of committing an *inverse crime*, that is, obtaining a great reconstruction due to the fact that the simulated data resonates in some helpful way with the reconstruction algorithm. We explain how this can occur and how to avoid it.

The use of Besov spaces and wavelets as a means of regularization is included in addition to the classical methods of truncated singular value decomposition, Tikhonov regularization, and total variation regularization. Also, this is the first book to discuss the recently introduced sparsity-based parameter choice rules. Many practical problems demand the use of very large data sets, and appropriate large-scale variants of the above reconstruction methods are addressed as well.

Part I requires knowledge of basic linear algebra and matrix computations, some knowledge of PDEs, basic analysis, and some programming skills. Some material for Part I is provided in the appendices.

Part II addresses nonlinear inversion, and it is a suitable text for a graduate course in applied mathematics. Also, we have received many requests for a text on the D-bar method, and Part II is designed to fulfill this need. We hope that researchers in electrical impedance tomography (EIT) will welcome this exposition.

The guiding example for Part II is EIT, although several other examples are also discussed briefly. Taking one guiding example is in accordance with the above rationale of treating nonlinear inverse problems as unique cases needing tailored solution strategies. We hope that the detailed discussion of the regularized D-bar method for EIT serves as a model for further research regarding other nonlinear inverse problems.

Actually, we see the regularized D-bar method as a topic that combines and unifies several schools of thought. Namely, there appear to be rather separated research traditions in the field of inverse problems, including the following three traditions: (1) The *analytic-geometric tradition* treats inverse problems as coefficient recovery tasks for PDEs. The main questions studied are uniqueness and stability proofs for recovering coefficient functions from limited but infinite-precision information. (2) The *regularization tradition* studies the construction of continuous maps from the data space to the model space providing approximate reconstructions from indirect measurement data containing errors. The main questions studied are convergence rates of the reconstructions at the asymptotic limit of zero measurement noise. (3) The *engineering tradition* is involved with writing robust computational algorithms that recover useful information from practical data.

In the regularized D-bar method, the reconstruction technique is defined using complex geometric optics solutions and a nonlinear Fourier transform introduced by the analytic-geometric tradition, the regularization strategy is provided by a nonlinear low-pass filtering step that can be analyzed according to the standards of the regularization tradition, and the final result is a robust imaging algorithm applicable to *in vivo* medical EIT.

Part II requires some background in PDEs, complex analysis, functional analysis, the finite element method (FEM), and numerical solution methods for linear systems. However, we do include short introductions to these topics in the text and in the appendices.

Throughout the book we provide exercises and project works involving MATLAB programming. Selected pieces of software can be downloaded from the website www.siam.org/books/cs10.

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