# Appendix

# Complementary Material

## A.2 Coprime Polynomials

***Proof of Theorem* A.2.5 [1].** Lemma A.2.4 guarantees the existence of polynomials $c(s)$ and $d(s)$ such that

$$a(s)c(s)+b(s)d(s)=1. \tag{AC.1}$$

Multiplying both sides of (AC.1) by the polynomial $a^*(s)$, we obtain

$$a^*(s)a(s)c(s)+a^*(s)b(s)d(s)=a^*(s). \tag{AC.2}$$

Dividing $a^*(s)d(s)$ by $a(s)$, i.e.,

$$\frac{a^*(s)d(s)}{a(s)}=r(s)+\frac{p(s)}{a(s)},$$

where $r(s)$ is the quotient of degree $n_{a^*}+n_d-n_a$, $p(s)$ is the remainder of degree $n_p<n_a$, and $n_{a^*},n_a,n_d$ are the degrees of $a^*(s),a(s),d(s)$, respectively, we have

$$a^*(s)d(s)=r(s)a(s)+p(s). \tag{AC.3}$$

Substituting (AC.3) into (AC.2), we obtain

$$l(s)a(s)+p(s)b(s)=a^*(s), \tag{AC.4}$$

where $l(s)=a^*(s)c(s)+r(s)b(s)$. (AC.4) implies that $\deg(l(s)a(s))=\deg(a^*(s)-p(s)b(s))\le\max\{n_{a^*},n_p+n_b\}$. Hence, $n_l\triangleq\deg(l(s))\le\max\{n_{a^*}-n_a,n_p+n_b-n_a\}$. Therefore, we have established the existence of polynomials $l(s)$ and $p(s)$ of degree $n_l\le\max\{n_{a^*}-n_a,n_p+n_b-n_a\}$ and $n_p<n_a$, respectively, that satisfy (AC.4). Since $n_p<n_a$, we also have $n_l\le\max\{n_a^*-n_a,n_b-1\}$.

Next, we show the uniqueness of $l(s)$ and $p(s)$. Suppose that $(l_1(s),p_1(s)),(l_2(s),p_2(s))$ are two solutions of (AC.4) that satisfy the degree constraints $n_p<n_a,n_l\le\max\{n_{a^*}-n_a,n_b-1\}$, i.e.,

$$a(s)l_1(s)+b(s)p_1(s)=a^*(s),a(s)l_2(s)+b(s)p_2(s)=a^*(s).$$

Subtracting one equation from the other, we have

$$a(s)(l_1(s)-l_2(s))+b(s)(p_1(s)-p_2(s))=0,$$

which implies that

$$\frac{b(s)}{a(s)}=\frac{l_2(s)-l_1(s)}{p_1(s)-p_2(s)}. \tag{AC.5}$$

Since $n_p < n_a$, (AC.5) implies that $b(s), a(s)$ have common factors, which contradicts the assumption that $a(s)$ and $b(s)$ are coprime. Thus, $l_1(s)=l_2(s)$ and $p_1(s)=p_2(s)$; i.e., the solution $l(s)$, $p(s)$ of (AC.4), is unique. □

***Proof of Theorem* A.2.6 [1].**
**If** Consider the polynomial equation

$$a(s)c(s)+b(s)d(s)=1, \tag{AC.6}$$

where $c(s)=c_{n-1}s^{n-1}+c_{n-2}s^{n-2}+\cdots+c_0$, $d(s)=d_{n-1}s^{n-1}+d_{n-2}s^{n-2}+\cdots+d_0$. Equating the coefficients of equal powers of $s$ on both sides of (AC.6), we obtain

$$S_e p=[0,0,\ldots,0,1]^T, \tag{AC.7}$$

where $p=[c_{n-1},c_{n-2},\ldots,c_0,d_{n-1},d_{n-2},\ldots,d_0]^T$.

Since $S_e$ is nonsingular, (AC.7) has a unique solution for $p$. Therefore, the solution of (AC.6) for $c(s)$ and $d(s)$ is also unique, which according to Lemma A.2.4 implies that $a(s), b(s)$ are coprime.

**Only if** Given that $a(s)$ and $b(s)$ are coprime, we prove nonsingularity of $S_e$ using contradiction. Suppose that $S_e$ is singular. Then

$$S_e x=0$$

has a nontrivial solution for $x$. Equivalently, there exist nonzero polynomials $p(s)$ and $q(s)$ of degree $n_p < n$ and $n_q < n$, respectively, such that

$$a(s)p(s)+b(s)q(s)=0$$

and hence

$$\frac{b(s)}{a(s)}=-\frac{p(s)}{q(s)}.$$

Since $n_p < n$ and $n_q < n$, $a(s)$ and $b(s)$ must have common factors, which contradicts the assumption that they are coprime. Hence, $S_e$ has to be nonsingular. □

## A.4 Properties of Functions and Matrices

***Proof of Lemma* A.4.5 [1].** (i) Since $f$ is bounded from below,

$$f_m = \inf_{0\le t\le\infty} f(t)$$

exists, which implies that there exists a sequence $\{t_n\}\in[0,\infty)$ such that $\lim_{n\to\infty} f(t_n) = f_m$. Therefore, given any $\varepsilon > 0$, there exists an integer $N > 0$ such that

$$|f(t_n) - f_m| < \varepsilon \quad \forall n \ge N.$$

Moreover, since $f$ is nonincreasing, for any $t \ge t_N$ we have

$$|f(t) - f_m| \le |f(t_N) - f_m| < \varepsilon.$$

Since $\varepsilon > 0$ is any given number, it follows that $\lim_{t\to\infty} f(t) = f_m$.

(ii) The proof is straightforward for $p = \infty$. For finite $p$, since $g \in \mathcal{L}_p$, we have

$$\left(\int_0^t f^p(\tau)d\tau\right)^{1/p} \le \left(\int_0^\infty g^p(\tau)d\tau\right)^{1/p} < \infty \quad \forall t \ge 0.$$

Moreover, since $\int_0^t f^p(\tau)d\tau$ is nondecreasing, we can establish, as in (i), that $\lim_{t\to\infty}(\int_0^t f^p(\tau)d\tau)^{1/p}$ exists and is finite, which implies that $f \in \mathcal{L}_p$. □

***Proof of Lemma* A.4.8 [1].** Assume that $\lim_{t\to\infty} f(t) \ne 0$. Then there exists an $\varepsilon > 0$ such that for every $T > 0$ one can find a sequence of numbers $t_i > T$ such that $|f(t_i)| > \varepsilon\ \forall i$. Since $f$ is uniformly continuous, there exists a number $\delta(\varepsilon) > 0$ such that

$$|f(t) - f(t_i)| < \frac{\varepsilon}{2} \quad \forall t \in [t_i, t_i + \delta(\varepsilon)].$$

Hence, for every $t \in [t_i, t_i + \delta(\varepsilon)]$, we have

$$|f(t)| \ge |f(t_i)| - |f(t) - f(t_i)| \ge \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2},$$

which implies that

$$\left|\int_{t_i}^{t_i+\delta(\varepsilon)} f(\tau)d\tau\right| = \int_{t_i}^{t_i+\delta(\varepsilon)} |f(\tau)|d\tau \ge \frac{\varepsilon\delta(\varepsilon)}{2}, \tag{AC.8}$$

noting that $f(t)$ retains the same sign for $t \in [t_i, t_i + \delta(\varepsilon)]$. On the other hand, existence of $\lim_{t\to\infty} \int_0^t f(\tau)d\tau$ implies that there exists a $T(\varepsilon) > 0$ such that for any $t_2 > t_1 > T(\varepsilon)$ we have

$$\left| \int_{t_1}^{t_2} f(\tau)d\tau \right| < \frac{\varepsilon\delta(\varepsilon)}{2},$$

which for $t_1 = t_i, t_2 = t_i + \delta(\varepsilon)$ contradicts (AC.8). Therefore, $\lim_{t\to\infty} f(t)$ has to be zero. □

## A.5 I/O Stability

***Proof of Corollary* A.5.6 [1].** Since $H(s)$ is proper and analytic in $\mathrm{Re}[s] \geq 0$, it can be expressed as

$$H(s) = H_0 + H_s(s),$$

where $H_0$ is a constant and $H_s(s)$ is strictly proper and analytic in $\mathrm{Re}[s] \geq 0$. Writing the I/O relation in the form

$$y = H_s u + y_s,$$
$$y_s = H_s(s)u,$$

we see using Corollary A.5.5 that $y_s \in \mathcal{L}_2 \bigcap \mathcal{L}_\infty$ and $|y_s(t)| \to 0$ as $t \to \infty$. Therefore, from $u \in \mathcal{L}_2 \bigcap \mathcal{L}_\infty$ and $u(t) \to 0$ as $t \to \infty$, it follows that $y \in \mathcal{L}_2 \bigcap \mathcal{L}_\infty$ and $|y(t)| \to 0$ as $t \to \infty$. □

***Proof of Corollary* A.5.8 [1].** Corollary A.5.5(i) indicates that

$$|y(t)| \leq \int_{t_0}^{t} |h(t-\tau)u(\tau)|\, d\tau \leq \int_{t_0}^{t} \alpha_1 e^{-\alpha_0(t-\tau)} |u(\tau)|\, d\tau \quad \forall t \geq t_0 \geq 0$$

for some constants $\alpha_1, \alpha_0 > 0$. Applying the Schwarz inequality, we obtain

$$|y(t)|^2 \leq \alpha_1^2 \int_{t_0}^{t} e^{-\alpha_0(t-\tau)} d\tau \int_{t_0}^{t} e^{-\alpha_0(t-\tau)} |u(\tau)|^2\, d\tau \leq \alpha_1^2 \alpha_0 \int_{t_0}^{t} e^{-\alpha_0(t-\tau)} |u(\tau)|^2\, d\tau. \qquad \text{(AC.9)}$$

Therefore, for any $t \geq t_0 \geq 0$ and $T \geq 0$ we have

$$\begin{aligned}\int_t^{t+T} |y(\tau)|^2\, d\tau &\le \frac{\alpha_1^2}{\alpha_0}\int_t^{t+T}\int_{t_0}^{\tau} e^{-\alpha_0(\tau-s)}\,|u(s)|^2\, ds d\tau \\ &= \frac{\alpha_1^2}{\alpha_0}\int_t^{t+T}\left(\int_{t_0}^{t} e^{-\alpha_0(\tau-s)}\,|u(s)|^2\, ds + \int_t^{\tau} e^{-\alpha_0(\tau-s)}\,|u(s)|^2\, ds\right)d\tau \\ &= \frac{\alpha_1^2}{\alpha_0}\int_t^{t+T} e^{-\alpha_0\tau}d\tau\int_{t_0}^{t} e^{\alpha_0 s}\,|u(s)|^2\, ds + \frac{\alpha_1^2}{\alpha_0}\int_t^{t+T} e^{\alpha_0 s}\,|u(s)|^2\left(\int_s^{t+T} e^{-\alpha_0\tau}d\tau\right)ds \\ &\le \frac{\alpha_1^2}{\alpha_0^2}\left(\int_{t_0}^{t} e^{-\alpha_0(t-s)}\,|u(s)|^2\, ds + \int_t^{t+T}|u(s)|^2\, ds\right).\end{aligned}$$

Since $u \in \mathcal{S}(\mu)$, it follows that

$$\int_t^{t+T} |y(\tau)|^2\, d\tau \le \frac{\alpha_1^2}{\alpha_0^2}\left[\Delta(t,t_0) + c_0\mu T + c_1\right], \tag{AC.10}$$

where

$$\begin{aligned}\Delta(t,t_0) &\triangleq \int_{t_0}^{t} e^{-\alpha_0(t-s)}\,|u(s)|^2\, ds \le e^{-\alpha_0 t}\sum_{i=0}^{n_t}\int_{i+t_0}^{i+1+t_0} e^{\alpha_0 s}\,|u(s)|^2\, ds \\ &\le e^{-\alpha_0 t}\sum_{i=0}^{n_t} e^{\alpha_0(i+1+t_0)}\int_{i+t_0}^{i+1+t_0}|u(s)|^2\, ds\end{aligned} \tag{AC.11}$$

and $n_t$ is an integer that depends on $t$ and satisfies $n_t + t_0 \le t < n_t + 1 + t_0$. Since $u \in \mathcal{S}(\mu)$, we have

$$\Delta(t,t_0) \le e^{-\alpha_0 t}(c_0\mu + c_1)\sum_{i=0}^{n_t} e^{\alpha_0(i+1+t_0)} \le \frac{c_0\mu + c_1}{1-e^{-\alpha_0}}e^{\alpha_0}. \tag{AC.12}$$

Substituting (AC.12) into (AC.10), we get

$$\int_t^{t+T} |y(\tau)|^2\, d\tau \le \frac{\alpha_1^2}{\alpha_0^2}\left(c_0\mu T + c_1 + \frac{c_0\mu + c_1}{1-e^{-\alpha_0}}e^{\alpha_0}\right) \quad \forall t \ge t_0 \ge 0,$$

i.e., $y \in \mathcal{S}(\mu)$. The upper bound for $|y(t)|^2$ can be calculated using (AC.9), (AC.11), (AC.12). $\square$

***Proof of Lemma* A.5.9 [1].** The transfer function $H(s)$ can be expressed as $H(s) = H_0 + H_s(s)$, and the corresponding impulse responses can be written as

$$h(t) = \begin{cases} 0, & t < 0, \\ H_0\delta_\Delta(t) + h_s(t), & t \ge 0, \end{cases}$$

where $\delta_\Delta(t)$ denotes the unit impulse function and $h(t), h_s(t)$ are the impulse responses of $H(s), H_s(s)$, respectively. Since $H(s)$ and hence $H_s(s)$ is analytic in $\mathrm{Re}[s] \geq -\delta/2$, we have $h_s \in \mathcal{L}_1$. Therefore, Remark A.5.3 applies to $H(s)$ and $h(t)$. Defining

$$h_\delta(t) \triangleq \begin{cases} 0, & t < 0, \\ H_0 \delta_\Delta(t) + e^{\frac{\delta}{2}t} h_s(t), & t \geq 0, \end{cases}$$
$$y_\delta(t) \triangleq e^{\frac{\delta}{2}t} y(t),$$
$$u_\delta(t) \triangleq e^{\frac{\delta}{2}t} u(t),$$

it follows from (A.16) that

$$y_\delta(t) = \int_0^t e^{\frac{\delta}{2}(t-\tau)} h(t-\tau) e^{\frac{\delta}{2}\tau} u(\tau) d\tau.$$

Since $u \in \mathcal{L}_{2e} \Rightarrow u_\delta \in \mathcal{L}_{2e}$ and $H(s-\delta/2)$ is the Laplace transform of $h_\delta$, applying Lemma A.5.2 and Remark A.5.3 for the signals $y_{\delta t}, u_{\delta t}$ at time $t$, we obtain

$$\|y_{\delta t}\|_2 \leq \|H(s-\delta/2)\|_\infty \|u_{\delta t}\|_2. \tag{AC.13}$$

Substituting $e^{-\frac{\delta}{2}t} \|y_{\delta t}\|_2 = \|y_t\|_{2\delta}$, $e^{-\frac{\delta}{2}t} \|u_{\delta t}\|_2 = \|u_t\|_{2\delta}$, and $\|H(s-\delta/2)\|_\infty = \|H(s)\|_{\infty\delta}$ into (AC.13), we complete the proof of (i).

In (ii), assuming that $H(s)$ is strictly proper we have

$$| y(t) | \leq \left| \int_0^t e^{\frac{\delta}{2}(t-\tau)} h(t-\tau) e^{-\frac{\delta}{2}(t-\tau)} u(\tau) d\tau \right|.$$

Applying the Schwarz inequality and Parseval's theorem [2], we obtain

$$| y(t) | \leq \left( \int_0^\infty e^{\delta(t-\tau)} | h(t-\tau) |^2 \, d\tau \right)^{1/2} \|u_t\|_{2\delta} = \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^\infty | H\left(j\omega - \frac{\delta}{2}\right) |^2 \, d\omega \right)^{1/2} \|u_t\|_{2\delta},$$

which completes the proof of (ii).

The last inequality can be shown by observing that

$$\begin{aligned}
\|H(s)\|_{2\delta} &= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty}\left|H\left(j\omega-\frac{\delta}{2}\right)\right|^2 d\omega\right)^{1/2} \\
&= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty}\left|\left(j\omega+p-\frac{\delta}{2}\right)H\left(j\omega-\frac{\delta}{2}\right)\right|^2 \frac{1}{|j\omega+p-\delta/2|^2}d\omega\right)^{1/2} \\
&\le \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty}\frac{1}{|j\omega+p-\delta/2|^2}d\omega\right)^{1/2}\sup_{\omega}\left(\left|\left(j\omega+p-\frac{\delta}{2}\right)H\left(j\omega-\frac{\delta}{2}\right)\right|\right) \\
&= \frac{1}{\sqrt{2(p-\delta/2)}}\left\|\left(s+p-\frac{\delta}{2}+\frac{\delta}{2}\right)H(s)\right\|_{\infty\delta}. \qquad \square
\end{aligned}$$

***Proof of Lemma* A.5.10 [1].** The solution of (A.20) for $x(t)$ is given by

$$x(t)=\Phi(t,0)x_0+\int_0^t\Phi(t,\tau)B(\tau)u(\tau)d\tau.$$

Therefore, using (A.21), we have

$$\begin{aligned}
|x(t)| &\le \|\Phi(t,0)\||x_0|+\int_0^t\|\Phi(t,\tau)\|\|B(\tau)\||u(\tau)|d\tau \\
&\le \varepsilon_t+c\lambda_0\int_0^t e^{-\alpha_0(t-\tau)}|u(\tau)|d\tau.
\end{aligned} \tag{AC.14}$$

Applying the Schwarz inequality to $\int_0^t e^{-\alpha_0(t-\tau)}|u(\tau)|d\tau = \int_0^t e^{-(2\alpha_0-\delta)(t-\tau)/2}e^{-\delta(t-\tau)/2}|u(\tau)|d\tau$ in (AC.14), we obtain

$$|x(t)|\le \varepsilon_t+c\lambda_0\left(\int_0^t e^{-(2\alpha_0-\delta)(t-\tau)}d\tau\right)^{1/2}\left(\int_0^t e^{-\delta(t-\tau)}|u(\tau)|^2\,d\tau\right)^{1/2}\le \varepsilon_t+\frac{c\lambda_0}{\sqrt{2\alpha_0-\delta}}\|u_t\|_{2\delta},$$

which completes the proof of (i).

By definition of the $\mathcal{L}_{2\delta}$ norm, it follows from (AC.14) that

$$\|x_t\|_{2\delta}\le\|\varepsilon_t\|_{2\delta}+c\lambda_0\|Q_t\|_{2\delta},$$

where

$$\|Q_t\|_{2\delta}\triangleq\left\|\left(\int_0^t e^{-\alpha_0(t-\tau)}|u(\tau)|d\tau\right)_t\right\|_{2\delta}=\left[\int_0^t e^{-\delta(t-\tau)}\left(\int_0^\tau e^{-\alpha_0(\tau-s)}|u(s)|ds\right)^2d\tau\right]^{1/2}.$$

Using the Schwarz inequality, we have

$$\left(\int_0^\tau e^{-\alpha_0(\tau-s)}\,|u(s)|\,ds\right)^2 = \left(\int_0^\tau e^{-\left(\alpha_0-\frac{\delta_1}{2}\right)(\tau-s)} e^{-\frac{\delta_1}{2}(\tau-s)}\,|u(s)|\,ds\right)^2$$
$$\le \int_0^\tau e^{-(2\alpha_0-\delta_1)(\tau-s)}ds \int_0^\tau e^{-\delta_1(\tau-s)}\,|u(s)|^2\,ds$$
$$\le \frac{1}{2\alpha_0-\delta_1}\int_0^\tau e^{-\delta_1(\tau-s)}\,|u(s)|^2\,ds.$$

Therefore,

$$\|Q_t\|_{2\delta} \le \frac{1}{\sqrt{2\alpha_0-\delta_1}}\left(\int_0^t e^{-\delta(t-\tau)}\int_0^\tau e^{-\delta_1(\tau-s)}\,|u(s)|^2\,ds d\tau\right)^{1/2}$$
$$= \frac{1}{\sqrt{2\alpha_0-\delta_1}}\left(\int_0^t e^{-\delta t+\delta_1 s}\,|u(s)|^2 \int_s^t e^{-(a_1-\delta)\tau}d\tau ds\right)^{1/2}$$
$$= \frac{1}{\sqrt{2\alpha_0-\delta_1}}\left(\int_0^t e^{-\delta t+\delta_1 s}\,|u(s)|^2\,\frac{e^{-(\delta_1-\delta)s}-e^{-(\delta_1-\delta)t}}{\delta_1-\delta}ds\right)^{1/2}$$
$$= \frac{1}{\sqrt{2\alpha_0-\delta_1}}\left(\int_0^t e^{-\delta(t-s)}-e^{-\delta_1(t-s)}\delta_1-\delta\,|u(s)|^2\,ds\right)^{1/2}$$
$$\le \frac{1}{\sqrt{(2\alpha_0-\delta_1)(\delta_1-\delta)}}\left(\int_0^t e^{-\delta(t-s)}\,|u(s)|^2\,ds\right)^{1/2},$$

and (ii) follows using the fact that $\|\varepsilon_t\|_{2\delta} \le \varepsilon_t$. (iii) can be proven using (ii) and the inequality

$$\|y_t\|_{2\delta} \le \left\|(C^T x)_t\right\|_{2\delta} + \|(Du)_t\|_{2\delta} \le \left(\sup_t\|C(t)\|\right)\|x_t\|_{2\delta} + \left(\sup_t\|D(t)\|\right)\|u_t\|_{2\delta}. \qquad \square$$

***Proof of Lemma* A.5.12 [1].** By definition of UCO, it can be shown that there exists a matrix $K(t)$ with bounded continuous elements such that the state-transition matrix $\Phi_c(t,\tau)$ of $A_c(t) \triangleq A(t)-K(t)C^T(t)$ satisfies

$$\|\Phi_c(t,\tau)\| \le \lambda_1 e^{-\alpha_1(t-\tau)}$$

for some constant $\lambda_1 > 0$. Rewriting (A.20) as

$$\dot{x} = (A-KC^T)x + Bu + KC^T x$$
$$= A_c(t)x + \bar{B}u + Ky,$$

where $\bar{B} \triangleq B-KD$, and following the procedure used in the proof of Lemma A.5.10, we obtain

$$|x(t)| \le \frac{\lambda_1}{\sqrt{2\alpha_1-\delta_0}}\left(c_1\|u_t\|_{2\delta_0} + c_2\|y_t\|_{2\delta_0}\right) + \varepsilon_t$$

and

$$\|x_t\|_{2\delta_0} \le \frac{\lambda_1}{\sqrt{(\delta_1-\delta_0)(2\alpha_1-\delta_1)}}\left(c_1\|u_t\|_{2\delta_0}+c_2\|y_t\|_{2\delta_0}\right)+\varepsilon_t,$$

where $c_1=\sup_t \|\bar{B}(t)\|, c_2=\sup_t \|K(t)\|$, and $\varepsilon_t$ is a term exponentially decaying to zero due to $x(0)=x_0$. Hence, we have proven (i) and (ii). The proof of (iii) follows directly from (A.20). □

***Proof of Lemma* A.5.13 [1].** (i) From (A.22) we have

$$\begin{aligned} x(t) &= e^{A(t-t_1)}x(t_1)+\bar{x}(t), \\ y(t) &= C^T e^{A(t-t_1)}x(t_1)+\bar{y}(t) \end{aligned} \tag{AC.15}$$

$\forall\, t\ge t_1$ where $\bar{x}(t),\bar{y}(t)$ are defined for $t\ge 0$ as

$$\begin{aligned} \bar{x}(t) &\triangleq \int_0^t e^{A(t-\tau)}Bv(\tau)d\tau, \\ \bar{y}(t) &\triangleq \int_0^t C^T e^{A(t-\tau)}Bv(\tau)d\tau + Dv(t), \\ v(t) &\triangleq \begin{cases} 0 & \text{if } \ t<t_1, \\ u(t) & \text{if } \ t\ge t_1. \end{cases} \end{aligned} \tag{AC.16}$$

Note that $\bar{x}$ , $\bar{y}$ in (AC.16) is the solution of the system

$$\begin{aligned} \dot{\bar{x}} &= A\bar{x}+Bv, \quad \bar{x}(0)=0, \\ \bar{y} &= C^T\bar{x}+Dv, \end{aligned} \tag{AC.17}$$

whose transfer function is $H(s)=C^T(sI-A)^{-1}B+D$. Since $A$ is a stable matrix, there exist constants $\lambda_0$ , $\alpha_0>0$ such that

$$\left\|e^{A(t-\tau)}\right\| \le \lambda_0 e^{-\alpha_0(t-\tau)},$$

i.e., $H(s)$ is analytic in $Re[s]\ge -\alpha_0$. Applying Lemma A.5.10 to (AC.17), we obtain

$$\begin{aligned} |\,\bar{x}(t)\,| &\le \frac{\lambda_0\|B\|}{\sqrt{2\alpha_0-\delta}}\|v_t\|_{2\delta} = c_0\|B\|\|v_t\|_{2\delta}, \\ \|\bar{x}_t\|_{2\delta} &\le \frac{\lambda_0\|B\|}{\sqrt{(\delta_1-\delta)(2\alpha_0-\delta_1)}}\|v_t\|_{2\delta} = c_1\|v_t\|_{2\delta}. \end{aligned}$$

Since $\|v_t\|_{2\delta}=\left\|u_{t,t_1}\right\|_{2\delta}$ and $\left\|\bar{x}_{t,t_1}\right\|_{2\delta}\le\|\bar{x}_t\|_{2\delta}$, it follows that for all $t\ge t_1$

$$|\,\bar{x}(t)\,| \le c_0\|B\|\left\|u_{t,t_1}\right\|_{2\delta}, \qquad \left\|\bar{x}_{t,t_1}\right\|_{2\delta} \le c_1\left\|u_{t,t_1}\right\|_{2\delta}. \tag{AC.18}$$

Substituting (AC.18) into (AC.15), we get

$$
\begin{aligned}
|x(t)| &\le \lambda_0 e^{-\alpha_0(t-t_1)}|x(t_1)| + |\bar{x}(t)| \\
&\le \lambda_0 e^{-\alpha_0(t-t_1)}|x(t_1)| + c_0\|B\|\|u_{t,t_1}\|_{2\delta} \quad \forall t \ge t_1.
\end{aligned}
$$

(ii) Using (AC.15), we have

$$
\begin{aligned}
\|x_{t,t_1}\|_{2\delta} &\le \left\|\left(e^{A(t-t_1)}x(t_1)\right)_{t,t_1}\right\|_{2\delta} + \|\bar{x}_{t,t_1}\|_{2\delta} \\
&\le \left(\int_{t_1}^{t} e^{-\delta(t-\tau)}e^{-2\alpha_0(\tau-t_1)}d\tau\right)^{1/2}\lambda_0|x(t_1)| + \|\bar{x}_{t,t_1}\|_{2\delta} \\
&\le \frac{\lambda_0 e^{-\delta(t-t_1)/2}}{\sqrt{2\alpha_0-\delta}}|x(t_1)| + \|\bar{x}_{t,t_1}\|_{2\delta}.
\end{aligned} \tag{AC.19}
$$

Substituting (AC.18) into (AC.19), the result follows.

(iii) Applying the results of Lemma A.5.9 to the system (AC.17), we have

$$
\|\bar{y}_t\|_{2\delta} \le \|H(s)\|_{\infty\delta}\|v_t\|_{2\delta}.
$$

Since $\|v_t\|_{2\delta} = \|u_{t,t_1}\|_{2\delta}$ and $\|\bar{y}_{t,t_1}\|_{2\delta} \le \|\bar{y}_t\|_{2\delta}$, we have

$$
\|\bar{y}_{t,t_1}\|_{2\delta} \le \|H(s)\|_{\infty\delta}\|u_{t,t_1}\|_{2\delta}. \tag{AC.20}
$$

From (AC.15) we have

$$
|y(t)| \le \|C\|\lambda_0 e^{-\alpha_0(t-t_1)}|x(t_1)| + |\bar{y}(t)| \quad \forall t \ge t_1, \tag{AC.21}
$$

which implies that

$$
\|y_{t,t_1}\|_{2\delta} \le \|C\|\frac{\lambda_0}{\sqrt{2\alpha_0-\delta}}e^{-\delta(t-t_1)/2}|x(t_1)| + \|\bar{y}_{t,t_1}\|_{2\delta} \quad \forall t \ge t_1,
$$

and the result follows immediately.

(iv) For $H(s)$ strictly proper, we have

$$
|\bar{y}(t)| \le \|H(s)\|_{2\delta}\|v_t\|_{2\delta} = \|H(s)\|_{2\delta}\|u_{t,t_1}\|_{2\delta} \quad \forall t \ge t_1. \tag{AC.22}
$$

Using (AC.22) and (AC.21), we establish the result. □

## A.6 Bellman–Gronwall Lemma

***Proof of Lemma* A.6.1 [1].** Because $k(t)$ is nonnegative, we have

$$
q(t) \triangleq k(t)e^{-\int_{t_0}^{t} g(\tau)k(\tau)d\tau} \ge 0 \quad \forall t \ge t_0.
$$

Multiplying both sides of (A.24) by $q(t)$ and rearranging the inequality, we obtain

$$q(t)y(t)-q(t)g(t)\int_{t_0}^{t} k(s)y(s)ds=\frac{d}{dt}\left(e^{-\int_{t_0}^{t} g(\tau)k(\tau)d\tau}\int_{t_0}^{t} k(s)y(s)ds\right)\le\lambda(t)q(t). \qquad \text{(AC.23)}$$

Integrating (AC.23), we obtain

$$e^{-\int_{t_0}^{t} g(\tau)k(\tau)d\tau}\int_{t_0}^{t} k(s)y(s)ds\le\int_{t_0}^{t}\lambda(s)q(s)ds.$$

Therefore,

$$\begin{aligned}\int_{t_0}^{t} k(s)y(s)ds &\le e^{\int_{t_0}^{t} g(\tau)k(\tau)d\tau}\int_{t_0}^{t}\lambda(s)q(s)ds\\ &= e^{\int_{t_0}^{t} g(\tau)k(\tau)d\tau}\int_{t_0}^{t}\lambda(s)k(s)e^{-\int_{t_0}^{s} g(\tau)k(\tau)d\tau}ds \qquad \text{(AC.24)}\\ &= \int_{t_0}^{t}\lambda(s)k(s)e^{\int_{s}^{t} g(\tau)k(\tau)d\tau}ds,\end{aligned}$$

and (A.25) follows by substituting (AC.24) into (A.24).

For the special case where $\lambda$ is a constant and $g=1$, from (A.24) we have

$$q_1\triangleq\lambda+\int_{t_0}^{t} k(s)y(s)ds\ge y(t).$$

Since $k\ge 0$, we have

$$\dot{q}_1=ky\le kq_1.$$

Defining $w\triangleq\dot{q}_1-kq_1\le 0$ and noting that $q_1(t_0)=\lambda$, we obtain

$$y(t)\le q_1(t)=e^{\int_{t_0}^{t} k(\tau)d\tau}q_1(t_0)+\int_{t_0}^{t} e^{\int_{\tau}^{t} k(s)ds}w(\tau)d\tau\le\lambda e^{\int_{t_0}^{t} k(\tau)d\tau}. \qquad \square$$

***Proof of Lemma* A.6.2 [1].** Defining $z(t)\triangleq\lambda(t)+\int_{t_0}^{t} k(s)y(s)ds$ and $v(t)\triangleq z(t)-y(t)$, we have

$$\dot{z}=\dot{\lambda}+ky=kz+\left(\dot{\lambda}-kv\right),\qquad z(t_0)=\lambda(t_0),$$

whose solution is given by

$$z(t)=e^{\int_{t_0}^{t} k(s)ds}z(t_0)+\int_{t_0}^{t} e^{\int_{\tau}^{t} k(s)ds}\left(\dot{\lambda}(\tau)-k(\tau)v(\tau)\right)d\tau.$$

Since $k(\tau), v(\tau) \geq 0$, we have

$$\int_{t_0}^{t} e^{\int_{\tau}^{t} k(s)ds} k(\tau)v(\tau)d\tau \geq 0$$

and hence

$$\begin{aligned} y(t) \leq z(t) &\leq e^{\int_{t_0}^{t} k(s)ds} z(t_0) + \int_{t_0}^{t} e^{\int_{\tau}^{t} k(s)ds} \dot{\lambda}(\tau)d\tau \\ &= \lambda(t_0)e^{\int_{t_0}^{t} k(s)ds} + \int_{t_0}^{t} \dot{\lambda}(s)e^{\int_{s}^{t} k(\tau)d\tau} ds. \qquad \square \end{aligned}$$

***Proof of Lemma* A.6.3 [1].** Rewriting the inequality

$$y(t) \leq c_0 e^{-\alpha(t-t_0)} + c_1 + c_2 \int_{t_0}^{t} e^{-\alpha(t-\tau)} k(\tau)y(\tau)d\tau$$

as

$$\bar{y}(t) \leq \lambda(t) + \int_{t_0}^{t} \bar{k}(\tau)\bar{y}(\tau)d\tau,$$

where $\bar{y}(t) = e^{\alpha t}y(t), \bar{k}(t) = c_2 k(t), \lambda(t) = c_0 e^{\alpha t_0} + c_1 e^{\alpha t}$, and applying B–G Lemma 2, we obtain

$$e^{\alpha t}y(t) \leq (c_0 + c_1)e^{\alpha t_0} e^{c_2 \int_{t_0}^{t} k(s)ds} + c_1\alpha \int_{t_0}^{t} e^{\alpha\tau} e^{c_2 \int_{\tau}^{t} k(s)ds} d\tau.$$

Multiplying each side of this inequality by $e^{-\alpha t}$, the result follows. $\square$

## A.8 Stability of Linear Systems

***Proof of Lemma* A.8.4 [1] (if part).** By definition of UCO, the if part is equivalent to the following: If there exist positive constants $\beta_1, \beta_2 > 0$ such that the observability Gramian $N(t_0, t_0 + \nu)$ of the system $(C, A)$ satisfies

$$\beta_1 I \leq N(t_0, t_0 + \nu) \leq \beta_2 I, \tag{AC.25}$$

then the observability Gramian $\bar{N}(t_0, t_0 + \nu)$ of $(C, A + KC^T)$ satisfies

$$\bar{\beta}_1 I \leq \bar{N}(t_0, t_0 + \nu) \leq \bar{\beta}_2 I \tag{AC.26}$$

for some constant $\bar{\beta}_1, \bar{\beta}_2 > 0$. Consider the following systems corresponding to (AC.25) and (AC.26), respectively:

$$\dot{x} = Ax,\ y = C^T x, \tag{AC.27}$$

$$\begin{aligned} \dot{\overline{x}} &= (A + KC^T)\overline{x}, \\ \overline{y} &= C^T \overline{x}. \end{aligned} \tag{AC.28}$$

From (AC.27), (AC.28) we have that $e \triangleq \overline{x} - x$ satisfies

$$\dot{e} = Ae + KC^T \overline{x}. \tag{AC.29}$$

Assuming that $\overline{x}(t_0) = x(t_0)$ and denoting the state-transition matrix of (AC.29) by $\Phi$, this implies that

$$e(t) = \int_{t_0}^{t} \Phi(t,\tau)K(\tau)C^T(\tau)\overline{x}(\tau)d\tau.$$

Applying the Schwarz inequality, we obtain

$$\begin{aligned} \left|C^T(t)e(t)\right|^2 &\le \int_{t_0}^{t} \left|C^T(t)\Phi(t,\tau)K(\tau)C^T(\tau)\overline{x}(\tau)\right|^2 d\tau \\ &\le \int_{t_0}^{t} \left|C^T(t)\Phi(t,\tau)\overline{x}_1(\tau)\right|^2 |K(\tau)|^2\ d\tau \int_{t_0}^{t} \left|C^T(\tau)\overline{x}(\tau)\right|^2 d\tau, \end{aligned} \tag{AC.30}$$

where

$$\overline{x}_1 \triangleq \begin{cases} KC^T\overline{x}\big/\left|KC^T\overline{x}\right| \text{ if } |C^T\overline{x}| \ne 0, \\ K/|K| \text{ if } |C^T\overline{x}| = 0. \end{cases}$$

Using the inequality $(a+b)^2 \le 2a^2 + 2b^2$ and the UCO property of $(C, A)$, this further implies that

$$\begin{aligned} &\int_{t_0}^{t_0+\nu} |C^T(t)\overline{x}(t)|^2\ dt \le 2\int_{t_0}^{t_0+\nu} |C^T(t)x(t)|^2\ dt + 2\int_{t_0}^{t_0+\nu} |C^T(t)e(t)|^2\ dt \\ &\le 2\int_{t_0}^{t_0+\nu} |C^T(t)x(t)|^2\ dt + 2\int_{t_0}^{t_0+\nu}\int_{t_0}^{t} \left|C^T(t)\Phi(t,\tau)\overline{x}_1(\tau)\right|^2 |K(\tau)|^2\ d\tau \int_{t_0}^{t} \left|C^T(\tau)\overline{x}(\tau)\right|^2 d\tau dt \\ &\le 2\beta_2\,|\overline{x}(t_0)|^2 + 2\int_{t_0}^{t_0+\nu}\int_{t_0}^{t} \left|C^T(t)\Phi(t,\tau)\overline{x}_1(\tau)\right|^2 |K(\tau)|^2\ d\tau \int_{t_0}^{t} \left|C^T(\tau)\overline{x}(\tau)\right|^2 d\tau dt. \end{aligned}$$

Applying B–G Lemma 2 (Lemma A.6.2), we obtain

$$\begin{aligned} \int_{t_0}^{t_0+\nu} \left|C^T(t)\overline{x}(t)\right|^2 dt &\le 2\beta_2 \left|\overline{x}(t_0)\right|^2 e^{2\int_{t_0}^{t_0+\nu}\int_{t_0}^{t} \left|C^T(t)\Phi(t,\tau)\overline{x}_1(\tau)\right|^2 |K(\tau)|^2 d\tau dt} \\ &= 2\beta_2 \left|\overline{x}(t_0)\right|^2 e^{2\int_{t_0}^{t_0+\nu}\int_{\tau}^{t_0+\nu} \left|C^T(t)\Phi(t,\tau)\overline{x}_1(\tau)\right|^2 dt |K(\tau)|^2 d\tau}. \end{aligned} \tag{AC.31}$$

Since $\left|\bar{x}_1\right| = 1$, (AC.25) implies that

$$\int_{t_0}^{t_0+\nu} \left|C^T(t)\Phi(t,\tau)\bar{x}_1(\tau)\right|^2 dt \le \int_{t_0}^{t_0+\nu} \left|C^T(t)\Phi(t,\tau)\right|^2 dt \le \beta_2 \tag{AC.32}$$

for any $t_0 \le \tau \le t_0 + \nu$. Since $\int_{t_0}^{t_0+\nu} |K(\tau)|^2\, d\tau \le k_\nu$, (AC.31), (AC.32) lead to

$$\int_{t_0}^{t_0+\nu} |C^T(t)\bar{x}(t)|^2\, dt \le 2\beta_2 e^{2\beta_2 k_\nu}\, |\bar{x}(t_0)|^2 . \tag{AC.33}$$

Moreover, using the equalities $\bar{x} = x + e$, $x(t_0) = \bar{x}(t_0)$, the inequality $(a+b)^2 \ge \frac{1}{2}a^2 - b^2$, and the UCO property of $(C, A)$, we have

$$\begin{aligned}\int_{t_0}^{t_0+\nu} \left|C^T(t)\bar{x}(t)\right|^2 dt &\ge \frac{1}{2}\int_{t_0}^{t_0+\nu} \left|C^T(t)x(t)\right|^2 dt - \int_{t_0}^{t_0+\nu} \left|C^T(t)e(t)\right|^2 dt \\ &\ge \frac{\beta_1}{2}|\bar{x}(t_0)|^2 - \int_{t_0}^{t_0+\nu} \left|C^T(t)e(t)\right|^2 dt.\end{aligned}$$

Again, since $\int_{t_0}^{t_0+\nu} |K(\tau)|^2\, d\tau \le k_\nu$, using (AC.30), (AC.32), this further implies that

$$\begin{aligned}\int_{t_0}^{t_0+\nu} \left|C^T(t)\bar{x}(t)\right|^2 dt &\ge \frac{\beta_1}{2}\left|\bar{x}(t_0)\right|^2 - \left(\int_{t_0}^{t_0+\nu} \left|C^T(t)\bar{x}(t)\right|^2 dt\right) \int_{t_0}^{t_0+\nu}\int_{t_0}^{t} \left|C^T(t)\Phi(t,\tau)\bar{x}_1(\tau)\right|^2 \left|K(\tau)\right|^2 d\tau dt \\ &\ge \frac{\beta_1}{2}|\bar{x}(t_0)|^2 - \beta_2 k_\nu \int_{t_0}^{t_0+\nu} |C^T(t)\bar{x}(t)|^2\, dt,\end{aligned}$$

which can be rewritten as

$$\int_{t_0}^{t_0+\nu} |C^T(t)\bar{x}(t)|^2\, dt \ge \frac{\beta_1}{2(1+\beta_2 k_\nu)}|\bar{x}(t_0)|^2 . \tag{AC.34}$$

Combining (AC.33) and (AC.34), we have

$$\bar{\beta}_1 |x_1(t_0)|^2 \le \int_{t_0}^{t_0+\nu} |C^T(t)x_1(t)|^2\, dt \le \bar{\beta}_2 |x_1(t_0)|^2,$$

where $\bar{\beta}_1 = \frac{\beta_1}{2(1+\beta_2 k_\nu)} > 0$, $\bar{\beta}_2 = 2\beta_2 e^{2\beta_2 k_\nu} > 0$. Using the definition of the observability Gramian matrix, this further implies that (AC.26) holds and hence $(C, A + KC^T)$ is UCO. Thus the proof of the if part is complete.

The proof for the only if part is exactly the same as the if part since $(C, A)$ can be obtained from $(C, A + KC^T)$ using the output injection $A = (A + KC^T) - KC^T$. □

***Proof of Lemma* A.8.5 [1].** By definition of UCO, the hypothesis is equivalent to the statement that the observability Gramian

$$N(t,t+T)\triangleq\int_t^{t+T}\Phi^T(\tau,t)\bar{C}\bar{C}^T\Phi(\tau,t)d\tau$$

satisfies

$$\beta I\geq N(t,t+T)\geq\alpha I \tag{AC.35}$$

for some constant $\alpha,\beta>0$, where $\Phi(t,t_0)$ is the state-transition matrix of the system in the lemma. Since $\phi\in\mathcal{L}_\infty$ and $A$ is a stable matrix, $\Phi(t,t_0)$ is bounded, and hence the left inequality in (AC.35) follows. The right inequality in (AC.35) is equivalent to

$$\int_t^{t+T}y_0^2(\tau)d\tau\geq\alpha\left(|x_1(t)|^2+|x_2(t)|^2\right)=\alpha\left(|x_1(t)|^2+|x_2(0)|^2\right). \tag{AC.36}$$

From the system equation of the lemma, we have

$$y_0(\tau)=C^Tx_1(\tau)=C^Te^{A(\tau-t)}x_1(t)-\int_t^{\tau}C^Te^{A(\tau-\sigma)}B\bar{\phi}^T(\sigma)d\sigma x_2 \quad \forall\tau\geq t.$$

For simplicity, define $\bar{x}_1(\tau)\triangleq C^Te^{A(\tau-t)}x_1(t)$, $\bar{x}_2(\tau)\triangleq-\int_t^{\tau}C^Te^{A(\tau-\sigma)}B\phi^T(\sigma)d\sigma x_2$. Then, we have

$$\int_t^{t+T}y_0^2(\tau)d\tau\geq\int_t^{t+T_1}\frac{\bar{x}_1^2(\tau)}{2}d\tau-\int_t^{t+T_1}\bar{x}_2^2(\tau)d\tau+\int_{t+T_1}^{t+T}\frac{\bar{x}_2^2(\tau)}{2}d\tau-\int_{t+T_1}^{t+T}\bar{x}_1^2(\tau)d\tau \tag{AC.37}$$

for any $0<T_1<T$. Stability of $A$ implies that

$$|\bar{x}_1(\tau)|\leq k_1e^{-\gamma_1(\tau-t)}|x_1(t)|$$

for some $k_1,\gamma_1>0$, and, therefore,

$$\int_{t+T_1}^{t+T}\bar{x}_1^2(\tau)d\tau\leq\frac{k_1^2}{2\gamma_1}e^{-2\gamma_1T_1}|x_1(t)|^2. \tag{AC.38}$$

Furthermore, since $(C,A)$ is observable, we have

$$\int_t^{t+T_1}e^{A^T(t-\tau)}CC^Te^{A(t-\tau)}d\tau\geq k_2I$$

for any $T_1>T_2$ and some constants $k_2,T_2>0$, which implies that

$$\int_t^{t+T_1}\bar{x}_1^2(\tau)d\tau\geq k_2|x_1(t)|^2. \tag{AC.39}$$

Using the fact that $\alpha_2 n_2 T_0 I \geq \int_{t+T_1}^{t+T} \overline{\phi}\overline{\phi}^T d\tau \geq n_1 \alpha_1 T_0 I$ , where $n_1, n_2$ are the largest and smallest integers, respectively, that satisfy $n_1 \leq \frac{T-T_1}{T_0} \leq n_2$ , we obtain

$$\int_t^{t+T_1} \overline{x}_2^2(\tau) d\tau \leq \alpha_2 \left(T_0 + T_1\right) \left|x_2(0)\right|^2 \text{ and } \int_{t+T_1}^{t+T} \overline{x}_2^2(\tau) d\tau \geq \alpha_1 \left(T - T_1 - T_0\right) \left|x_2(0)\right|^2 .$$

These two inequalities together with (AC.37)–(AC.39) imply that

$$\int_t^{t+T} y_0^2(\tau) d\tau \geq \left(\frac{k_2}{2} - \frac{k_1^2}{2\gamma_1} e^{-2\gamma_1 T_1}\right) \left|x_1(t)\right|^2 + \left(\frac{\alpha_1}{2}\left(T - T_1 - T_0\right) - \alpha_2\left(T_1 + T_0\right)\right) \left|x_2(0)\right|^2 .$$

Choosing $T_1, T$ such that $T_1 > T_2$ , $\frac{k_2}{2} - \frac{k_1^2}{2\gamma_1} e^{-2\gamma_1 T_1} \geq \frac{k_2}{4}$ , $\frac{\alpha_1}{2}(T - T_1 - T_0) - \alpha_2 (T_1 + T_0) \geq \beta_1$ for some constant $\beta_1$ , we reach (AC.36) and the proof is complete. □

***Proof of Theorem*** **A.8.8 [1].** It follows from $Re\{\lambda_i(A(t))\} \leq -\sigma_s \ \forall t \geq 0$ and Theorem A.8.7 that the Lyapunov equation

$$A^T(t)P(t) + P(t)A(t) = -I$$

has a unique bounded solution $P(t)$ for each fixed $t$ , and $\dot{P}$ satisfies

$$A^T(t)\dot{P}(t) + \dot{P}(t)A(t) = -Q(t) \ \ \forall t \geq 0,$$

where $Q(t) = \dot{A}^T(t)P(t) + P(t)\dot{A}(t)$ . Since $Re\{\lambda_i(A(t))\} \leq -\sigma_s \ \forall t \geq 0$ , it can be verified that [3] the unique $\dot{P}$ is given by

$$\dot{P}\left(t\right) = \int_0^\infty e^{A^T(t)\tau} Q\left(t\right) e^{A(t)\tau} d\tau$$

and therefore

$$\left\|\dot{P}\left(t\right)\right\| \leq \left\|Q\left(t\right)\right\| \int_0^\infty \left\|e^{A^T(t)\tau}\right\| \left\|e^{A(t)\tau}\right\| d\tau.$$

Since $Re\{\lambda_i(A(t))\} \leq -\sigma_s \ \forall t \geq 0$ implies that $\| e^{A(t)\tau} \| \leq \alpha_1 e^{-\alpha_0 \tau}$ for some $\alpha_1, \alpha_0 > 0$ it follows that

$$\left\|\dot{P}(t)\right\| \leq c\left\|Q(t)\right\| \leq 2c\left\|P(t)\right\| \left\|\dot{A}(t)\right\|$$

for some $c \geq 0$ . This together with $P \in \mathcal{L}_\infty$ implies that

$$\left\|\dot{P}(t)\right\| \leq \beta \left\|\dot{A}(t)\right\| \ \forall t \geq 0 \qquad \text{(AC.40)}$$

for some constant $\beta \geq 0$ . For the Lyapunov function

$$V(t,x) = x^T P(t)x$$

we have

$$\dot{V} = -|x(t)|^2 + x^T(t)\dot{P}(t)x(t). \qquad \text{(AC.41)}$$

Substituting (AC.40) into (AC.41) and noting that $P$ satisfies $0 < \beta_1 \leq \lambda_{min}(P) \leq \lambda_{max}(P) \leq \beta_2$ for some $\beta_1, \beta_2 > 0$, we have

$$\dot{V}(t) \leq -|x(t)|^2 + \beta\left\|\dot{A}(t)\right\||x(t)|^2 \leq -\beta_2^{-1}V(t) + \beta\beta_1^{-1}\left\|\dot{A}(t)\right\|V(t)$$

and therefore

$$V(t) \leq e^{-\int_{t_0}^{t}(\beta_2^{-1} - \beta\beta_1^{-1}\|\dot{A}(\tau)\|)d\tau} V(t_0). \qquad \text{(AC.42)}$$

If condition (a) holds, from (AC.42) we have

$$V(t) \leq e^{-(\beta_2^{-1} - \beta\beta_1^{-1}\mu)(t-t_0)} e^{\beta\beta_1^{-1}\alpha_0} V(t_0)$$

and hence $\forall \mu \in [0, \mu^*)$, $\mu^* = \frac{\beta_1}{\beta_2\beta}$, $V(t) \to 0$ exponentially fast, which implies that $x_e = 0$ is u.a.s. in the large.

If (b) holds, we rewrite (AC.42) as

$$V(t) \leq e^{-\beta_2^{-1}(t-t_0)} e^{\beta\beta_1^{-1}\int_{t_0}^{t}\|\dot{A}(\tau)\|d\tau} V(t_0),$$

and by applying the Schwarz inequality we obtain

$$\begin{aligned}
\int_{t_0}^{t}\left\|\dot{A}(\tau)\right\|d\tau &\leq \left(\int_{t_0}^{t}\left\|\dot{A}(\tau)\right\|^2 d\tau\right)^{\frac{1}{2}}\sqrt{t-t_0} \\
&\leq \left[\mu^2(t-t_0)^2 + \alpha_0(t-t_0)\right]^{\frac{1}{2}} \\
&\leq \mu(t-t_0) + \sqrt{\alpha_0}\sqrt{t-t_0}.
\end{aligned}$$

Therefore,

$$V(t) \leq e^{-\alpha(t-t_0)} y(t) V(t_0),$$

where $\alpha = (1-\gamma)\beta_2^{-1} - \beta\beta_1^{-1}\mu$, $\gamma$ is an arbitrary constant that satisfies $0 < \gamma < 1$, and

$$\begin{aligned} y(t) &= \exp\left[-\gamma\beta_2^{-1}(t-t_0)+\beta\beta_1^{-1}\sqrt{\alpha_0}\sqrt{t-t_0}\right] \\ &= \exp\left[-\gamma\beta_2^{-1}\left(\sqrt{t-t_0}-\frac{\beta\beta_1^{-1}\sqrt{\alpha_0}}{2\gamma\beta_2^{-1}}\right)^2+\frac{\alpha_0\beta^2\beta_2}{4\gamma\beta_1^2}\right] \\ &\le \exp\left[\alpha_0\frac{\beta^2\beta_2}{4\gamma\beta_1^2}\right] \triangleq c \quad \forall t\ge t_0. \end{aligned}$$

Therefore, we have

$$V(t)\le ce^{-\alpha(t-t_0)}V(t_0).$$

Choosing $\mu^*=\frac{\beta_1(1-\gamma)}{\beta_2\beta}$, we have that for all $\mu\in[0,\mu^*)$, $\alpha>0$ and hence $V(t)\to 0$ exponentially fast, which implies that $x_e=0$ is u.a.s. in the large.

(c) is a special case of (a). □

## A.11 Swapping Lemmas

***Proof of Lemma* A.11.1 [1].** We have

$$\begin{aligned} W(s)\left[\theta^T\omega\right] &= W(s)\left[\omega^T\theta\right] = d\theta^T\omega + C^T\int_0^t e^{A(t-\tau)}B\omega^T\theta d\tau \\ &= d\theta^T\omega + C^Te^{At}\left(\int_0^\tau e^{-A\sigma}B\omega^T(\sigma)d\sigma\theta(\tau)\Bigg|_{\tau=0}^{\tau=t} - \int_0^t\int_0^\tau e^{-A\sigma}B\omega^T(\sigma)d\sigma\dot{\theta}(\tau)d\tau\right) \quad \text{(AC.43)} \\ &= \theta^T\left(d\omega + C^T\int_0^t e^{A(t-\sigma)}B\omega(\sigma)d\sigma\right) - C^T\int_0^t e^{A(t-\tau)}\int_0^\tau e^{A(\tau-\sigma)}B\omega^T(\sigma)d\sigma\dot{\theta}(\tau)d\tau. \end{aligned}$$

Since $C^T\int_0^t e^{A(t-\tau)}f(\tau)d\tau = C^T(sI-A)^{-1}f\Big|_t$ for an arbitrary differentiable bounded function $f(t)$, we have

$$\begin{aligned} &\theta^T\left(d\omega + C^T\int_0^t e^{A(t-\sigma)}B\omega(\sigma)d\sigma\right) - C^T\int_0^t e^{A(t-\tau)}\int_0^\tau e^{A(\tau-\sigma)}B\omega^T(\sigma)d\sigma\dot{\theta}(\tau)d\tau \\ &= \theta^T\left(d+C^T(sI-A)^{-1}\right)\omega - C^T(sI-A)^{-1}\left[\int_0^t e^{A(t-\tau)}B\omega^T(\tau)d\tau\dot{\theta}(t)\right], \end{aligned}$$

and hence (AC.43) can be rewritten as

$$W(s)\left[\theta^T\omega\right] = \theta^TW(s)\left[\omega\right] + W_c(s)\left[\left(W_b(s)\left[\omega^T\right]\right)\dot{\theta}\right]. \quad \square$$

***Proof of Lemma*** **A.11.2 [1].** We have

$$\begin{aligned}\theta^T\omega &= \left(1-F(s,\alpha_0)\right)\theta^T\omega + F(s,\alpha_0)\theta^T\omega \\ &= F_1(s,\alpha_0)s(\tilde{\theta}^T\omega) + F(s,\alpha_0)(\theta^T\omega) \\ &= F_1(s,\alpha_0)(\tilde{\theta}^T\omega + \theta^T\dot{\omega}) + F(s,\alpha_0)(\theta^T\omega),\end{aligned}$$

and hence the equality in the lemma is established. In order to establish the bound on $\|F_1(s,\alpha_0)\|_{\infty\delta}$ , we form a decomposition of

$$F_1(s,\alpha_0) = \frac{1-F(s,\alpha_0)}{s} = \frac{(s+\alpha_0)^k - \alpha_0^k}{s(s+\alpha_0)^k} \tag{AC.44}$$

using the equality

$$(s+\alpha_0)^k = \sum_{i=0}^{k} C_k^i s^i \alpha_0^{k-i} = \alpha_0^k + \sum_{i=1}^{k} C_k^i s^i \alpha_0^{k-i} = \alpha_0^k + s\sum_{i=1}^{k} C_k^i s^{i-1}\alpha_0^{k-i}, \tag{AC.45}$$

where $C_k^i \triangleq \frac{k!}{i!(k-i)!}$ ( $0! \triangleq 1$ ). Substituting (AC.45) into (AC.44), we obtain

$$F_1(s,\alpha_0) = \frac{\sum_{i=1}^k C_k^i s^{i-1}\alpha_0^{k-i}}{(s+\alpha_0)^k} = \frac{1}{s+\alpha_0}\sum_{i=1}^{k} C_k^i \frac{s^{i-1}}{(s+\alpha_0)^{i-1}} \frac{\alpha_0^{k-i}}{(s+\alpha_0)^{k-i}}.$$

Since $\alpha_0 > \delta$ ,

$$\left\| \frac{\alpha_0^i}{(s+\alpha_0)^i} \right\|_{\infty\delta} = \left( \left\| \frac{\alpha_0}{s+\alpha_0} \right\|_{\infty\delta} \right)^i = \left( \frac{2\alpha_0}{2\alpha_0-\delta} \right)^i \le 2^i, \quad i \ge 1,$$

and

$$\left\| \frac{1}{s+\alpha_0} \right\|_{\infty\delta} = \frac{2}{2\alpha_0-\delta} \le \frac{2}{\alpha_0}, \quad \left\| \frac{s^{i-1}}{(s+\alpha_0)^{i-1}} \right\|_{\infty\delta} = 1, \quad i \ge 1,$$

we have

$$\|F_1(s,\alpha_0)\|_{\infty\delta} \le \left\| \frac{1}{s+\alpha_0} \right\|_{\infty\delta} \sum_{i=1}^{k} C_k^i \left\| \frac{s^{i-1}}{(s+\alpha_0)^{i-1}} \right\|_{\infty\delta} \left\| \frac{\alpha_0^{k-i}}{(s+\alpha_0)^{k-i}} \right\| = \frac{\sum_{i=1}^k C_k^i 2^{k-i+1}}{2\alpha_0-\delta} \le \frac{c}{\alpha_0},$$

where $c \triangleq \sum_{i=1}^k C_k^i 2^{k-i+1}$ is independent of $\alpha_0$ . □

***Proof of Lemma* A.11.3 [1].** (i) follows immediately observing that

$$\begin{aligned}
&A(s,t)(B(s,t)f) = a_0 B(s,t)f + \sum_{i=1}^{n} a_i \sum_{j=0}^{m} s^i \left(b_j s^j f\right) \\
&= a_0 B(s,t)f + \sum_{i=1}^{n} a_i \sum_{j=0}^{m} s^{i-1} \left(b_j s^{j+1} f + \dot{b}_j s^j f\right) \\
&= a_0 B(s,t)f + \sum_{i=1}^{n} a_i \sum_{j=0}^{m} \left[s^{i-2} \left(b_j s^{j+2} f + \dot{b}_j s^{j+1} f\right) + s^{i-1} \dot{b}_j s^j f\right] \\
&\vdots \\
&= a_0 B(s,t)f + \sum_{i=1}^{n} a_i \sum_{j=0}^{m} \left(b_j s^{j+i} f + \sum_{k=0}^{i-1} s^k \left(\dot{b}_j s^{j+i-1-k} f\right)\right) \\
&= \sum_{i=0}^{n} a_i \sum_{j=0}^{m} b_j s^{j+i} f + \sum_{i=1}^{n} \sum_{j=0}^{m} \sum_{k=0}^{i-1} s^k \left(\dot{b}_j s^{j+i-1-k} f\right) = C(s,t)f + r(t),
\end{aligned}$$

where

$$\begin{aligned}
r(t) &\triangleq \sum_{i=1}^{n} a_i \sum_{k=0}^{i-1} \mathrm{s}^k \sum_{j=0}^{m} \left(\dot{b}_j \, \mathrm{s}^{j+i-1-k} f\right) \\
&= a^T D_{n-1}(s) r_b(t)
\end{aligned}$$

and

$$\begin{aligned}
r_b(t) &\triangleq \left[\sum_{j=0}^{m} \dot{b}_j s^{j+n-1} f, \ \sum_{j=0}^{m} \dot{b}_j s^{j+n-2} f, \dots, \sum_{j=0}^{m} \dot{b}_j s^j f\right]^T \\
&= \left(\begin{bmatrix} s^{n-1+m} & s^{n-1+m-1} & \cdots & s^{n-1} \\ s^{n-2+m} & s^{n-2+m-1} & \cdots & s^{n-2} \\ \vdots & \vdots & & \vdots \\ s^m & s^{m-1} & \cdots & 1 \end{bmatrix} f\right) \begin{bmatrix} \dot{b}_m \\ \dot{b}_{m-1} \\ \vdots \\ \dot{b}_0 \end{bmatrix} = \left(\alpha_{n-1}(s) \alpha_m^T(s) f\right) \dot{b}.
\end{aligned}$$

Using (i) and noting that there is no restriction on the polynomial orders $n, m$, we have

$$A(s,t)(B(s,t)f) = C(s,t)f + a^T D_{n-1}(s)\left(\left[\alpha_{n-1}(s)\alpha_m^T(s) f\right]\dot{b}\right) \tag{AC.46}$$

and

$$B(s,t)(A(s,t)f) = C(s,t)f + b^T D_{m-1}(s)\left(\left[\alpha_{m-1}(s)\alpha_n^T(s) f\right]\dot{a}\right). \tag{AC.47}$$

Subtracting (AC.47) from (AC.46), we obtain

$$\begin{aligned}
&A(s,t)(B(s,t)f) - B(s,t)(A(s,t)f) \\
&= a^T D_{n-1}(s)\left(\left[\alpha_{n-1}(s)\alpha_m^T(s) f\right]\dot{b}\right) - b^T D_{m-1}(s)\left(\left[a_{m-1}(s)\alpha_n^T(s) f\right]\dot{a}\right) \\
&= \left[a^T D_{n-1}(s), b^T D_{m-1}(s)\right]\left(\begin{bmatrix} 0 & \alpha_{n-1}(s)\alpha_m^T(s) \\ -\alpha_{m-1}(s)\alpha_n^T(s) & 0 \end{bmatrix}\begin{bmatrix} \dot{a} \\ \dot{b} \end{bmatrix}\right),
\end{aligned}$$

and the proof of (ii) is complete. □

***Proof of Lemma* A.11.4 [1].** (i) By definition, we have

$$A(s,t)(B(s,t)f) = \sum_{i=0}^{n}\sum_{j=0}^{m} s^i(a_i b_j s^j f),$$
$$\overline{B}(s,t)(A(s,t)f) = \sum_{i=0}^{n}\sum_{j=0}^{m} s^j(a_i b_j s^i f).$$

We examine $s^i(a_i b_j s^j f)$ for the three cases $i > j$, $i < j$, and $i = j$ separately.

***i* > *j*** We have

$$s^i(a_i b_j s^j f) = s^j\left(s^{i-j}(a_i b_j s^j f)\right).$$

Applying the equality $s(a_i b_j s^l f) = a_i b_j s^{l+1} f + (\dot{a}_i b_j + a_i \dot{b}_j) s^l f$ successively for $l = j, j+1, \ldots, i-1$, we obtain

$$s^{i-j}(a_i b_j s^j f) = a_i b_j s^i f + \sum_{k=1}^{i-j} s^{i-j-k}(\dot{a}_i b_j + a_i \dot{b}_j) s^{j-1+k} f$$

and hence

$$s^i(a_i b_j s^j f) = s^j(a_i b_j s^i f) + \sum_{k=1}^{i-j} s^{i-k}(\dot{a}_i b_j + a_i \dot{b}_j) s^{j-1+k} f, \quad i > j. \qquad \text{(AC.48)}$$

***i* < *j*** We have

$$s^i(a_i b_j s^j f) = s^i((a_i b_j s^{j-i}) s^i f).$$

Applying $a_i b_j s^l f = s(a_i b_j) s^{l-1} f - (\dot{a}_i b_j + a_i \dot{b}_j) s^{l-1} f$ successively for $l = j$ down to $i+1$, we obtain

$$(a_i b_j s^{j-i}) s^i f = s^{j-i}(a_i b_j) s^i f - \sum_{k=1}^{j-i} s^{j-i-k}(\dot{a}_i b_j + a_i \dot{b}_j) s^{i-1+k} f$$

and hence

$$s^i(a_i b_j s^j f) = s^j(a_i b_j s^i f) - \sum_{k=1}^{j-i} s^{j-k}(\dot{a}_i b_j + a_i \dot{b}_j) s^{i-1+k} f, \quad i < j. \qquad \text{(AC.49)}$$

***i* = *j*** We have

$$s^i(a_i b_j s^j f) = s^j(a_i b_j s^i f), \quad i = j. \qquad \text{(AC.50)}$$

Combining (AC.48), (AC.49), and (AC.50), we have

$$\overline{A}(s,t)(B(s,t)f) = \sum_{i=0}^{n}\sum_{j=0}^{m} s^j(a_i b_j s^i f) + r_1 = B(s,t)(A(s,t)f) + r_1,$$

where

$$r_1 \triangleq \sum_{i=0}^{n}\sum_{\substack{j=0\\j<i}}^{m}\sum_{k=1}^{i-j} s^{i-k}(\dot{a}_i b_j + a_i \dot{b}_j)s^{j-1+k}f - \sum_{i=0}^{n}\sum_{\substack{j=0\\j>i}}^{m}\sum_{k=1}^{j-i} s^{j-k}(\dot{a}_i b_j + a_i \dot{b}_j)s^{i-1+k}f.$$

Since all the $s$ terms in $r_1$ before and after $\dot{a}_i b_j + a_i \dot{b}_j$ have order less than $\max\{n,m\}-1$, and $r_1$ can be expressed as

$$r_1 = \alpha_{\bar{n}}^T(s)F(a,b)\alpha_{\bar{n}}(s)f,$$

where $\bar{n} = \max\{n,m\}-1$ and $F(a,b) \in \mathcal{R}^{\bar{n}\times\bar{n}}$ is a TV matrix whose elements are linear combinations of $\dot{a}_i b_j + a_i \dot{b}_j$, we establish the identity in (i). Since $a,b \in \mathcal{L}_\infty$, it also follows that

$$\|F(a,b)\| \le c_1 |\dot{a}| + c_2 |\dot{b}|.$$

(ii) Applying Swapping Lemma 1 to $\theta = a_i$ and $\omega = b_j$ with $W(s) = \frac{s^j}{\Lambda_0(s)}$, we get

$$a_i b_j \frac{s^j}{\Lambda_0(s)} f = \frac{s^j}{\Lambda_0(s)} a_i b_j f - W_{jc}((W_{jb} f)(\dot{a}_i b_j + a_i \dot{b}_j)),$$

where $W_{jc}(s), W_{jb}(s)$ are strictly proper transfer functions, having the same poles as $\frac{1}{\Lambda_0(s)}$. Therefore,

$$\begin{aligned}\bar{A}(s,t)\left(B(s,t)\frac{1}{\Lambda_0(s)}f\right) &= \sum_{i=0}^{n}\sum_{j=0}^{m} s^i\left(a_i b_j \frac{s^j}{\Lambda_0(s)}f\right)\\ &= \sum_{i=0}^{n}\sum_{j=0}^{m}\frac{s^{i+j}}{\Lambda_0(s)}(a_i b_j f) - \sum_{i=0}^{n}\sum_{j=0}^{m} s^i W_{jc}((W_{jb}f)(\dot{a}_i b_j + a_i \dot{b}_j))\\ &= \frac{1}{\Lambda_0(s)}\overline{A(s,t)\cdot B(s,t)}f + r_2,\end{aligned}$$

where

$$\begin{aligned}&r_2 \triangleq \alpha_n^T(s)G(s,f,a,b),\\ &G(s,f,a,b) \triangleq [g_n,\ldots,g_1,g_0], \quad g_j = -\sum_{j=0}^{m} W_{jc}(s)\big((W_{jb}f)(\dot{a}_i b_j + a_i \dot{b}_j)\big). \quad \square\end{aligned}$$

## A.12 Discrete-Time Systems

***Proof of Theorem* A.12.21.** (i) Given $k_0$ and $\varepsilon > 0$, we need to show that there exists $\delta(\varepsilon, k_0)$ which depends on $\varepsilon$ and $k_0$ such that $|x_0| < \delta$ implies $|x(k;k_0,x_0)| < \varepsilon$ $\forall k \ge k_0$. By definition we know that $V(k,0) = 0$ $\forall k \in \mathcal{Z}^+$ and there exists a continuous function $\phi: [0,\infty) \to [0,\infty)$ such that $V(k,x) \ge \phi(|x|)$ $\forall k \in \mathcal{Z}^+, x \in \mathcal{B}(r)$ and some $r > 0$. Without loss of generality we assume that $\varepsilon \le r$ (for $\varepsilon > r$, $\delta(r,k_0)$ can be used to bound $|x_0|$ from above). Given any $k_0, k \geq k_0$, and $0 < \varepsilon \leq r$, $|x(k;k_0,x_0)| < \varepsilon$ is quaranteed if

$$\phi\left(|x(k)|\right) < \phi(\varepsilon) \Leftarrow V(k,x) < \phi(\varepsilon).$$

According to (1), we have $\Delta V(k,x) \leq 0$ and hence $V(k,x(k)) \leq V(k_0,x_0)$, for any $k \geq k_0$. Therefore, $|x(k;k_0,x_0)| < \varepsilon$ is quaranteed if $V(k_0,x_0) < \phi(\varepsilon)$. Since $V(k_0,0) = 0$ and $V(k_0,x) \geq \phi(|x|)$ is continuous at $x = 0$, there exists a positive $\delta(\varepsilon,k_0)$ such that $|x_0| < \delta$ implies $V(k_0,x_0) < \phi(\varepsilon)$ and hence $|x(k;k_0,x_0)| < \varepsilon$.

(ii) Due to (i) we have that $x_e = 0$ is stable. (2) implies that $\phi(|x|) \leq V(k,x) \leq w(|x|)$ $\forall k \in \mathcal{Z}^+, x \in \mathcal{B}(r)$, where $\phi, w : [0,\infty) \to [0,\infty)$ are continuous strictly increasing functions. According to (i), $|x(k;k_0,x_0)| < \varepsilon$ is quaranteed if $V(k_0,x_0) < \phi(\varepsilon)$. Therefore, for $\delta(\varepsilon) = w^{-1}(\phi(\varepsilon))$, $|x_0| < \delta$ implies $V(k_0,x_0) \leq w(|x_0|) < \phi(\varepsilon)$ and hence $|x(k;k_0,x_0)| < \varepsilon$.

(iii) Due to (i) we have that $x_e = 0$ is stable. (3) implies that for any $k \geq k_0$ there exists some $k_1 \geq k$ such that $V(k_1,x(k_1)) < V(k,x(k))$. Hence $V(k,x(k))$ is a strictly decreasing function of $k$, and since $V(k,x) \geq \phi(|x|)$ is continuous at $x = 0$ and (a) $\Delta V(k,x) \leq -\bar{\phi}(|x|)$ is continuous at $x = 0$ or (b) since $\Delta V(k,x)$ is not identically zero along any trajectory other than $x \equiv 0$, where $\phi, \bar{\phi} : [0,\infty) \to [0,\infty)$ are continuous strictly increasing functions, $V(k,x)$ may converge only to 0. Therefore $x_e = 0$ is a.s.

(iv) follows directly using the arguments in (ii) and (iii).

(v) is a direct consequence of (ii) and Definition A.12.20.

(vi) is a direct consequence of (iii) and Definition A.12.20.

(vii) We need to show that there exists a $0 < \alpha < 1$, and for every $\beta > 0$ there exists $\gamma(\beta) > 0$ which is independent of $k$ such that

$$\left|x(k;k_0,x_0)\right| \leq \gamma(\beta)\alpha^{k-k_0}$$

$\forall k \geq k_0$ whenever $|x_0| < \beta$. Given $|x_0| < \beta$, we have, because of (5)(a), $c_1|x_0|^2 \leq V(k_0,x_0) < c_2\beta^2$. Furthermore, using (5)(b) and the right inequality of (5)(a), for any $k \geq k_0$ we obtain

$$V(k+1,x(k+1)) \leq V(k,x(k)) - c_3|x(k)|^2 \leq \left(1 - \frac{c_3}{c_2}\right)V(k,x(k)).$$

Therefore, for any $k \geq k_0$,

$$V(k,x(k)) \leq \left(1 - \frac{c_3}{c_2}\right)^{k-k_0} V(k_0,x_0) \leq \left(1 - \frac{c_3}{c_2}\right)^{k-k_0} c_2\beta^2.$$

Applying the left inequality of (5)(a), we get

$$\left|x(k;k_0,x_0)\right| \leq \gamma(\beta)\alpha^{k-k_0},$$

where $\gamma(\beta)=\frac{c_2}{c_0}\beta$ and $\alpha=\left(1-\frac{c_3}{c_2}\right)^{1/2}$.

(viii) is a direct consequence of (vii) and Definition A.12.20. □

***Proof of Theorem*** **A.12.24.** (i) From (A.59) we have that

$$x(k)=A^k x(0)+\sum_{j=0}^{k-1}A^j Bu(k-j-1) \quad \forall k\geq 0.$$

Therefore, using (1)–(3) and Theorem A.12.23, we obtain

$$|x(k)|\leq \gamma_1^k\left(|x(0)|\right)+\sum_{j=0}^{k-1}\gamma_2^j\left|u(k-j-1)\right| \quad \forall k\geq 0$$

for some constants $0<\gamma_1(|x(0)|),\gamma_2<1$. Hence,

$$\begin{aligned}\sum_{k=0}^{N}|x(k)|^2 &\leq \sum_{k=0}^{N}\left(\gamma_1^k+\sum_{j=0}^{k-1}\gamma_2^j\left|u(k-j-1)\right|\right)^2\\ &\leq 2\sum_{k=0}^{N}\gamma_1^{2k}+2\sum_{k=0}^{N}\sum_{j=0}^{k-1}\gamma_2^{2j}\left|u(k-j-1)\right|^2\\ &\leq \frac{2}{1-\gamma_1^2}+2\sum_{k=0}^{N-1}|u(k)|^2\sum_{j=k}^{N-1}\gamma_2^{2(j-k)}\\ &\leq k_1\sum_{k=0}^{N}|u(k)|^2+k_2,\end{aligned}$$

where $k_1=\frac{2}{1-\gamma_2^2}$, $k_2=\frac{2}{1-\gamma_1^2}$.

(ii) We have established above that

$$|x(k)|\leq \gamma_1^k\left(|x(0)|\right)+\sum_{j=0}^{k-1}\gamma_2^j\left|u(k-j-1)\right| \quad \forall k\geq 0.$$

Therefore,

$$|x(k)|\leq m_1+m_2\max_{0\leq\tau<N}|u(\tau)|,$$

where $m_1=\gamma_1$, $m_2=\frac{1}{1-\gamma_2}$. □

***Proof of Lemma*** **A.12.32.** (i) From (A.61), we can write the solution $x$ in the closed form as

$$x(k)=\Phi(k,k_0)x_0+\sum_{i=0}^{k-1}\Phi(k,i+1)f\left(i,x(i)\right). \tag{AC.51}$$

Since the unperturbed linear difference equation is e.s., we have

$$\left\|\Phi(k,i)\right\|\leq\beta\lambda^{k-i}$$

for some $\beta > 0$ and $0 \le \lambda < 1$. Using the property of $f(k,x)$, (AC.51) can be written as

$$\begin{aligned} \left|x(k)\right| &\le \beta\lambda^{k-k_0}\left|x_0\right| + \beta\lambda^{-1}\sum_{i=0}^{k-1}\lambda^{k-i}\left(\gamma_0(i)\left|x(i)\right| + \gamma_1(i)\right) \\ &\le \beta\left(\lambda^{k-k_0}\left|x_0\right| + \gamma\frac{1-\lambda^k}{1-\lambda}\right) + \beta\lambda^{-1}\sum_{i=k_0}^{k-1}\lambda^{k-i}\gamma_0(i)\left|x(i)\right|, \end{aligned} \tag{AC.52}$$

where $\gamma = \sup_i\{\gamma_1(i)\}$. Setting $s_k = \lambda^{-k}\,|\,x(k)\,|$, (AC.52) becomes

$$s_k \le h_k + \beta\lambda^{-1}\sum_{i=k_0}^{k-1}\gamma_0(i)s_i,$$

where $h_k = \beta(\lambda^{-k_0}\left|x_0\right| + \gamma(\lambda^{-k} - 1)/(1-\lambda))$. Using the discrete-time B–G lemma, we have

$$s_k \le h_k + \sum_{i=k_0}^{k-1}\left\{\left[\prod_{j=i+1}^{k-1}\left(1+\beta\lambda^{-1}\gamma_0(j)\right)\right]\beta\lambda^{-1}\gamma_0(i)h_i\right\},$$

which is equivalent to

$$\left|x(k)\right| \le \lambda^k h_k + \beta\sum_{i=k_0}^{k-1}\left\{\left[\prod_{j=i+1}^{k-1}\left(\lambda+\beta\gamma_0(j)\right)\right]\gamma_0(i)\lambda^i h_i\right\}.$$

Since $\gamma_0(k)$ satisfies $\frac{1}{N}\sum_{k=k_0}^{k_0+N-1}\gamma_0(k) \le \mu + \frac{a_0}{N}$, it can be shown that

$$\prod_{j=k}^{k+N-1}\left(\lambda+\beta\gamma_0(j)\right) \le \beta_1\bar{\mu}^N, \quad \bar{\mu} = \left(\lambda+\beta\nu\right)^{\left(1-\frac{\mu}{\nu}\right)}\beta_2^{\frac{\mu}{\nu}} \quad \forall\nu > 0,$$

where $\beta_1$ is a constant and $\beta_2$ is an upper bound of $\lambda + \beta\gamma_0(j)$ which is assumed to be greater than 1. Since $0 \le \lambda < 1$, choosing

$$\nu = \frac{1-\lambda}{2\beta}, \qquad \mu^* = \frac{1-\lambda}{2\beta}\frac{\ln 2\left(2+\lambda\right) - \ln 3\left(1+\lambda\right)}{\ln 2\beta_2 - \ln\left(1+\lambda\right)},$$

it can be shown that $\mu^* > 0$ and $\bar{\mu} \le (2+\lambda)/3$ for $\mu \in [0,\mu^*)$. Therefore, we have

$$\left|x(k)\right| \le \lambda^k h_k + \beta\beta_1\sum_{i=k_0}^{k-1}\left(\frac{2+\lambda}{3}\right)^{k-i-1}\gamma_0(i)\lambda^i h_i. \tag{AC.53}$$

Since $\gamma_1(k)$ is bounded, we have that $\lambda^k h_k$ and $\lambda^k h_k\gamma_0(k)$ are u.b. Therefore, from (AC.53), we can write

$$\left|x(k)\right| \le \alpha_1 + \alpha_2\frac{1-\left(\frac{2+\lambda}{3}\right)^{k-k_0+1}}{1-\frac{2+\lambda}{3}} \tag{AC.54}$$

for some constants $\alpha_{1,}\alpha_2 > 0$, and thus $|\,x(k)\,|$ is u.b.

(ii) If $\gamma_1(k)=0\ \forall k$, then $h_k=\beta|x_0|$ and (AC.53) can be written as

$$|x(k)|\leq \beta\lambda^k|x_0|+\alpha_3\left(\frac{3\lambda}{2+\lambda}\right)^{k_0}\left(\frac{2+\lambda}{3}\right)^{k-1}\frac{1-\left(\frac{3\lambda}{2+\lambda}\right)^{k-k_0}}{1-\frac{3\lambda}{2+\lambda}},$$

where $\alpha_3=\beta^2\beta_1\,|\,x_0\,|\sup_k\gamma_0(k)$. Since $0\leq\lambda<1$, we have $0\leq (2+\lambda)/3, 3\lambda/(2+\lambda)<1$; therefore, from (AC.54) it follows that as $k\to\infty$, $|x(k)|\to 0$ exponentially fast. □

***Proof of Lemma*** **A.12.36.** We have

$$\begin{aligned}\theta^T\omega &= \left(1-F(z,\alpha_0)\right)\theta^T\omega+F(z,\alpha_0)\theta^T\omega\\ &= F_1(z,\alpha_0)(z-1)(\theta^T\omega)+F(z,\alpha_0)(\theta^T\omega)\\ &= F_1(z,\alpha_0)(\Delta\theta^T\omega+\theta^T\Delta\omega+\Delta\theta^T\Delta\omega)+F(z,\alpha_0)(\theta^T\omega),\end{aligned}$$

and hence the equality in the lemma is established. In order to establish the bound on $\|F_1(z,\alpha_0)\|_{\infty\delta}$, we form a decomposition of

$$F_1(z,\alpha_0)=\frac{1-F(z,\alpha_0)}{z-1}=\frac{(z-1+\alpha_0)^k-\alpha_0^k}{(z-1)(z-1+\alpha_0)^k} \tag{AC.55}$$

using the equality

$$(z-1+\alpha_0)^k=\sum_{i=0}^{k}C_k^i(z-1)^i\alpha_0^{k-i}=\alpha_0^k+\sum_{i=1}^{k}C_k^i(z-1)^i\alpha_0^{k-i}=\alpha_0^k+(z-1)\sum_{i=1}^{k}C_k^i(z-1)^{i-1}\alpha_0^{k-i}, \tag{AC.56}$$

where $C_k^i\triangleq\frac{k!}{i!(k-i)!}$ ($0!\triangleq 1$). Substituting (AC.56) into (AC.55), we obtain

$$F_1(z,\alpha_0)=\frac{\sum_{i=1}^{k}C_k^i(z-1)^{i-1}\alpha_0^{k-i}}{(z-1+\alpha_0)^k}=\frac{1}{z-1+\alpha_0}\sum_{i=1}^{k}C_k^i\frac{(z-1)^{i-1}}{(z-1+\alpha_0)^{i-1}}\frac{\alpha_0^{k-i}}{(z-1+\alpha_0)^{k-i}}.$$

Since $\alpha_0\geq c_1(\sqrt{\delta}+1)$,

$$\left\|\frac{\alpha_0^i}{(z-1+\alpha_0)^i}\right\|_{\infty\delta}=\left(\left\|\frac{\alpha_0}{z-1+\alpha_0}\right\|_{\infty\delta}\right)^i=\left(\frac{\alpha_0}{\alpha_0-1-\sqrt{\delta}}\right)^i\leq d_1^i,\quad i\geq 1,$$

and

$$\left\|\frac{1}{z-1+\alpha_0}\right\|_{\infty\delta}=\frac{1}{\alpha_0-1-\sqrt{\delta}}\leq\frac{d_1}{\alpha_0},\qquad \left\|\frac{(z-1)^{i-1}}{(z-1+\alpha_0)^{i-1}}\right\|_{\infty\delta}=\frac{1}{(1+c_1)^{i-1}},\quad i\geq 1,$$

where $d_1\triangleq\left(\frac{c_1}{c_1-1}\right)$, we have

$$\left\|F_1(z,\alpha_0)\right\|_{\infty\delta} \le \left\|\frac{1}{z-1+\alpha_0}\right\|_{\infty\delta} \sum_{i=1}^{k} C_k^i \left\|\frac{(z-1)^{i-1}}{(z-1+\alpha_0)^{i-1}}\right\|_{\infty\delta} \left\|\frac{\alpha_0^{k-i}}{(z-1+\alpha_0)^{k-i}}\right\|$$
$$= \frac{d_1}{\alpha_0}\sum_{i=1}^{k} C_k^i \frac{d_1^{k-1}}{\left(1+c_1\right)^{i-1}} \le \frac{c}{\alpha_0},$$

where $c \triangleq \sum_{i=1}^{k} C_k^i \frac{d_1^k}{\left(1+c_1\right)^{i-1}}$ is independent of $\alpha_0$. □

# Bibliography

[1] P. A. IOANNOU AND J. SUN, *Robust Adaptive Control*, Prentice-Hall, Englewood Cliffs, NJ, 1996; also available at

http://www-rcf.usc.edu/~ioannou/Robust_Adaptive_Control.htm.

[2] C. A. DESOER, AND M. VIDYASAGAR, *Feedback Systems: Input-Output Properties*, Academic Press, New York, 1975.

[3] T. KAILATH, *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980.