

## Chapter 5

# Complementary Material

### 5.3.2 MRC Schemes: Known Plant Parameters

**Proof of Lemma 5.3.4.** For  $n = 1$ , we have  $\alpha(s) = 0$ ,  $\Lambda(s) = 1$ . Hence, (5.83) implies  $\theta_1^* = \theta_2^* = 0$  and a unique solution for  $\theta_3^*$ .

For  $n \geq 2$ , consider the monic greatest common divisor  $H_p(s)$  of  $Z_p(s)$  and  $R_p(s)$ . Then we can write  $Z_p(s) = \bar{Z}_p(s)H_p(s)$ ,  $R_p(s) = \bar{R}_p(s)H_p(s)$  for some coprime polynomial pair  $\bar{Z}_p(s), \bar{R}_p(s)$ . Since  $Z_p(s)$  is Hurwitz,  $H_p(s)$  also has to be Hurwitz. Hence we can divide both sides of (5.83) by  $H_p(s)$  and obtain

$$\theta_1^{*T} \alpha(s) \bar{R}_p(s) + k_p (\theta_2^{*T} \alpha(s) + \theta_3^* \Lambda(s)) \bar{Z}_p(s) = \Lambda(s) \bar{R}_p(s) - \bar{Z}_p(s) \Lambda_0(s) R_m(s).$$

It can be easily seen that the degree of the right-hand side is  $n_p + n - r_h - 2$ , where  $r_h$  is the degree of  $H_p(s)$ . Since  $\bar{Z}_p(s), \bar{R}_p(s)$  are coprime, Theorem A.2.5 (see the Appendix) implies that there exist unique polynomials  $a(s), b(s)$  of degree  $n - 2, n_p - r_h - 1$ , respectively, such that

$$a(s) \bar{R}_p(s) + b(s) \bar{Z}_p(s) = \Lambda(s) \bar{R}_p(s) - \bar{Z}_p(s) \Lambda_0(s) R_m(s).$$

Once the unique solution  $a(s), b(s)$  of the above equation is calculated, the general solution for  $\theta_1^{*T} \alpha(s), k_p (\theta_2^{*T} \alpha(s) + \theta_3^* \Lambda(s))$  can be obtained as

$$\begin{aligned} \theta_1^{*T} \alpha(s) &= \beta(s) \bar{Z}_p(s) + a(s), \\ k_p (\theta_2^{*T} \alpha(s) + \theta_3^* \Lambda(s)) &= -\beta(s) \bar{R}_p(s) + b(s), \end{aligned} \tag{5C.1}$$

where  $\beta(s)$  is any polynomial of degree  $n - n_p + r_h - 1$ . Once a polynomial  $\beta(s)$  of degree  $n - n_p + r_h - 1$  is fixed,  $\theta_1^*, \theta_2^*, \theta_3^*$  can be calculated equating the coefficients of both sides of (5C.1).

If  $R_p, Z_p$  are coprime (i.e.,  $r_h = 0$ ) and  $n = n_p$ , then  $f(s) = 0$  and hence the solution of (5C.1) for  $\theta_1^*, \theta_2^*, \theta_3^*$  is unique.  $\square$

## 5.4 Direct MRAC with Unnormalized Adaptive Laws

### 5.4.1 Relative Degree $n^* = 1$

*Proof of Theorem 5.4.1(ii) [1].* Since  $\dot{r} \in \mathcal{L}_\infty$ , from (i) we have that  $\omega, \dot{\omega} \in \mathcal{L}_\infty$ . Furthermore,  $\omega$  can be expressed as

$$\omega = \begin{bmatrix} c(sI - F)^{-1} g G_p^{-1}(s) y_p \\ (sI - F)^{-1} g y_p \\ y_p \\ r \end{bmatrix}.$$

Since  $y_p = y_m + e_1 = W_m(s)r + e_1$ , we have

$$\omega = H(s)r + \bar{\omega},$$

where  $\bar{\omega} = H_0(s)e_1$  and

$$H(s) = \begin{bmatrix} c(sI - F)^{-1} g G_p^{-1}(s) W_m(s) \\ (sI - F)^{-1} g W_m(s) \\ W_m(s) \\ 1 \end{bmatrix}, \quad H_0(s) = \begin{bmatrix} c(sI - F)^{-1} g G_p^{-1}(s) \\ (sI - F)^{-1} g \\ 1 \\ 0 \end{bmatrix}.$$

$H_0(s)$  is a proper transfer matrix whose poles are stable, and  $e_1 \in \mathcal{L}_2 \cap \mathcal{L}_\infty$  and goes to zero as  $t \rightarrow \infty$ . Hence, from Lemma 3C.1 we have  $\bar{\omega} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$  and  $|\bar{\omega}(t)| \rightarrow 0$  as  $t \rightarrow \infty$ . Therefore, by definition of PE,  $\omega$  is PE if  $H(s)r$  is PE. Since  $r$  is sufficiently rich of order  $2n$ , applying Theorem 3.4.3, this further implies that  $\omega$  is PE if  $\text{rank}[H(j\omega_1), H(j\omega_2), \dots, H(j\omega_{2n})] = 2n$  on  $\mathcal{C}^{2n}$  for any  $\omega_1, \omega_2, \dots, \omega_{2n} \in \mathcal{R}$  with  $\omega_i \neq \omega_j$  for  $i \neq j$ .  $H(s)$  can be rewritten as

$$H(s) = \frac{1}{k_p Z_p(s) \Lambda(s) R_m(s)} \begin{bmatrix} c\alpha(s) R_p(s) k_m Z_m(s) \\ \alpha(s) k_p Z_p(s) k_m Z_m(s) \\ \Lambda(s) k_p Z_p(s) k_m Z_m(s) \\ \Lambda(s) k_p Z_p(s) R_m(s) \end{bmatrix} = \frac{1}{D(s)} \bar{H} [s^l, s^{l-1}, \dots, 1]^T,$$

where  $D(s) = k_p Z_p(s) \Lambda(s) R_m(s)$ ,  $l = \deg(\Lambda(s) Z_p(s) R_m(s)) = 2n - 1 + q_m$ , and  $\bar{H} \in \mathcal{R}^{2n \times (l+1)}$  is a constant coefficient matrix. Therefore, we have

$$[H(j\omega_1), \dots, H(j\omega_{2n})] = \bar{H} \Psi(\omega_1, \dots, \omega_{2n}) \Upsilon(\omega_1, \dots, \omega_{2n}),$$

where

$$\Psi(\omega_1, \dots, \omega_{2n}) = \begin{bmatrix} (j\omega_1)^l & (j\omega_2)^l & \cdots & (j\omega_{2n})^l \\ (j\omega_1)^{l-1} & (j\omega_2)^{l-1} & \cdots & (j\omega_{2n})^{l-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\Upsilon(\omega_1, \dots, \omega_{2n}) = \text{diag} \left( \frac{1}{D(j\omega_1)}, \dots, \frac{1}{D(j\omega_{2n})} \right).$$

$\Upsilon(\omega_1, \dots, \omega_{2n})$  is obviously nonsingular.  $\Psi(\omega_1, \dots, \omega_{2n})$  is also nonsingular since it is a submatrix of the Vandermonde matrix and  $\omega_i \neq \omega_k, i \neq k$ . Therefore,  $\text{rank}[H(j\omega_1), \dots, H(j\omega_{2n})] = 2n$  if and only if  $\text{rank}(\bar{H}) = 2n$ . Next we show that  $\text{rank}(\bar{H}) = 2n$ . Assume to obtain the contradiction that  $\text{rank}(\bar{H}) < 2n$ . Then there exists a nonzero constant vector  $C \in \mathcal{R}^{2n}$  such that

$$C^T \bar{H} = 0.$$

Using the decomposition  $C = [C_1^T, C_2^T, c_3, c_4]^T$ , where  $C_1, C_2 \in \mathcal{R}^{n-1}; c_3, c_4 \in \mathcal{R}$ , this implies that

$$C^T \bar{H} [s^l, s^{l-1}, \dots, 1]^T = C_1^T \alpha(s) R_p(s) k_m Z_m(s) + C_2^T \alpha(s) k_p Z_p(s) k_m Z_m(s) + c_3 \Lambda(s) k_p Z_p(s) k_m Z_m(s) + c_4 \Lambda(s) R_m(s) k_p Z_p(s) = 0.$$

Observing that  $c_4$  is the coefficient of the  $s^l$  term, we have  $c_4 = 0$ . This further implies that

$$(C_1^T \alpha(s) R_p(s) + C_2^T \alpha(s) k_p Z_p(s) + c_3 \Lambda(s) k_p Z_p(s)) k_m Z_m(s) = 0$$

or, equivalently,

$$C_1^T \alpha(s) R_p(s) + (C_2^T \alpha(s) + c_3 \Lambda(s)) k_p Z_p(s) = 0,$$

i.e.,

$$\frac{k_p Z_p(s)}{R_p(s)} = \frac{-C_1^T \alpha(s)}{c_3 \Lambda(s) + C_2^T \alpha(s)}.$$

Since  $\deg(c_3 \Lambda(s) + C_2^T \alpha(s)) \leq n-1$ , this contradicts the coprimeness of  $Z_p(s), R_p(s)$ . Therefore,  $\text{rank}(\bar{H}) = 2n$  and hence  $\text{rank}[H(j\omega_1), \dots, H(j\omega_{2n})] = 2n$ . Thus we have shown that  $\omega$  is PE.

Now consider (5.98) and (5.100), i.e.,

$$\begin{aligned} \dot{e} &= A_c e + \bar{B}_c \rho^* \tilde{\theta}^T \omega, \quad e(0) = e_0, \\ \dot{\tilde{\theta}} &= -\Gamma e_1 \omega \text{sgn}(\rho^*), \\ e_1 &= C_c^T e, \end{aligned} \tag{5C.2}$$

and the Lyapunov-like function in (5.99), i.e.,  $V(\tilde{\theta}, e) = \frac{e^T P_c e}{2} + \frac{\tilde{\theta}^T \Gamma^{-1} \tilde{\theta}}{2} |\rho^*|$ . From (5.101) we have

$$\dot{V} = -\frac{e^T q q^T e}{2} - \frac{\nu_c}{2} e^T L_c e \leq -\nu_1 e^T C_c^T C_c e = -\nu_1 e_1^T e_1$$

for some constant  $\nu_1 > 0$ , noting that  $L_c > 0$ . We can rewrite (5C.2) as

$$\dot{\bar{e}} = A(t)\bar{e}, \quad e_1 = C^T \bar{e},$$

where

$$\bar{e} = \begin{bmatrix} e \\ \tilde{\theta} \end{bmatrix}, \quad A(t) = \begin{bmatrix} A_c & \bar{B}_c \rho^* \omega^T(t) \\ -\Gamma \omega(t) \text{sgn}(\rho^*) C_c^T & 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_c \\ 0 \end{bmatrix}.$$

We can also rewrite  $V(t)$  as  $V = \bar{e}^T P \bar{e}$ , where

$$P = \frac{1}{2} \begin{bmatrix} P_c & 0 \\ 0 & |\rho^*| \Gamma^{-1} \end{bmatrix}.$$

Hence we have  $\dot{V} = \bar{e}^T (PA + A^T P) \bar{e} \leq -\nu_1 e_1^T e_1 = -\nu_1 \bar{e}^T C C^T \bar{e}$ , which implies that  $PA(t) + A^T(t)P + \nu_1 C C^T$  is negative semidefinite. Therefore, using Theorem A.8.3, to show that  $\tilde{\theta}$  and  $e_1$  converge to zero asymptotically and hence exponentially fast, it is sufficient to show that  $(C, A)$  is a UCO pair.

By Lemma A.8.4,  $(C, A)$  is a UCO pair if and only if  $(C, A + KC^T)$  is a UCO pair, where

$$K(t) = \begin{bmatrix} 0 \\ -\text{sgn}(\rho^*) \Gamma \omega(t) \end{bmatrix},$$

which satisfies the integral condition in Lemma A.8.4 since  $\omega \in \mathcal{L}_\infty$ . The system corresponding to  $(C, A + KC^T)$  can be written as

$$\bar{e} = \begin{bmatrix} e \\ \tilde{\theta} \end{bmatrix}, \quad A(t) = \begin{bmatrix} A_c & \bar{B}_c \rho^* \omega^T(t) \\ -\Gamma \omega(t) \text{sgn}(\rho^*) C_c^T & 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_c \\ 0 \end{bmatrix},$$

$$\dot{x}_1 = Ax_1 + \bar{B}_c \rho^* \omega^T x_2,$$

$$\dot{x}_2 = 0,$$

$$y_0 = C_c^T x_1.$$

Observing that  $\rho^* C_c^T (sI - A_c)^{-1} \bar{B}_c$  is stable and minimum-phase, since  $\omega, \dot{\omega} \in \mathcal{L}_\infty$  and  $\omega$  is PE,  $\bar{\omega} = \rho^* C_c^T (sI - A_c)^{-1} \bar{B}_c \omega$  is PE as well because of Lemma 3C.1 (see Chapter 3:

Complementary Material). Therefore, using Lemma A.8.5, we have that  $(C, A + KC^T)$  is UCO, and the proof is complete.  $\square$

### 5.4.2 Relative Degree $n^* = 2$

**Proof of Theorem 5.4.3(ii) [1].** It has already been established that  $y_p, \phi \in \mathcal{L}_\infty$ . From (5.113) and the fact that  $\dot{e}_1 \in \mathcal{L}_\infty$  and hence  $\dot{y}_p \in \mathcal{L}_\infty$ , we have that  $\dot{\phi} \in \mathcal{L}_\infty$ . Next we show that  $\phi$  is PE. Similar to the case of  $n^* = 1$ , we write  $\phi$  as

$$\phi = H(s)r + \bar{\phi},$$

where  $\bar{\phi} = H_o(s)e_1$  and

$$H(s) = \frac{1}{s + p_0} \begin{bmatrix} (sI - F)^{-1} g G_p^{-1}(s) W_m(s) \\ (sI - F)^{-1} g W_m(s) \\ W_m(s) \\ 1 \end{bmatrix}, \quad H_o(s) = \frac{1}{s + p_0} \begin{bmatrix} (sI - F)^{-1} g G_p^{-1}(s) \\ (sI - F)^{-1} g \\ 1 \\ 0 \end{bmatrix}.$$

Since  $e_1 \in \mathcal{L}_2 \cap \mathcal{L}_\infty$  and  $\dot{e}_1 \in \mathcal{L}_\infty$ , it follows from Lemma 3C.1 that  $\bar{\phi} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$  and  $|\bar{\phi}| \rightarrow 0$  as  $t \rightarrow \infty$ . Hence we can apply exactly the same steps in the proof of Theorem 5.4.1 to show that  $\phi$  is PE.

Now consider the error equations (5.110), (5.112), which have the same form as (5.98), (5.100). As in the proof of Theorem 5.4.1(i), we use Lemma 3C.1 and the results above stating that  $\phi, \dot{\phi} \in \mathcal{L}_\infty$  and  $\phi$  is PE, and establish that the adaptive law (5.112) guarantees that  $|\tilde{\theta}|$  converges to zero exponentially fast. Using (5.110), this also implies that  $e_1$  converges to zero exponentially fast.  $\square$

### 5.4.3 Relative Degree Greater than 2

When the relative degree is greater than 2, the procedure is very similar but the control law becomes more complex. In this subsection, we present the relative degree-3 case to illustrate the approach and complexity.

When the relative degree is  $n^* = 3$ , the transfer function  $W_m(s)$  of the reference model cannot be chosen to be SPR because according to assumption M2 (stated in section 5.3.1), the relative degrees of the reference model  $W_m(s)$  and the plant  $G_p(s)$  should be the same. Therefore, the parameterization (5.103) and the control law  $u_p = \theta^T \omega$  will not lead to a desired error equation where the SPR characteristics can be employed. In order to be able to employ these characteristics, we rewrite (5.104) in a form that involves an SPR transfer function as follows:

$$e_1 = W_m(s)(s + p_0)(s + p_1)\rho^* (\bar{u}_p - \theta^{*T} \bar{\omega}),$$

where

$$\bar{u}_p = H_0(s)u_p, \quad \bar{\omega} = H_0(s)\omega, \quad H_0(s) = \frac{1}{(s+p_0)(s+p_1)},$$

and  $W_m(s), p_0, p_1$  are chosen so that  $\bar{W}_m(s) = W_m(s)/H_0(s) = W_m(s)(s+p_0)(s+p_1)$  with relative degree 1 is SPR. Without loss of generality let us consider

$$W_m(s) = \frac{1}{(s+p_0)(s+p_1)(s+p_2)}$$

for some  $p_2 > 0$  so that

$$e_1 = \frac{1}{s+p_2} \rho^* (\bar{u} - \theta^{*T} \bar{\omega}).$$

The estimate of  $\hat{e}_1$  of  $e_1$  is given by

$$\hat{e}_1 = \frac{1}{s+p_2} \rho (\bar{u}_p - \theta^T \bar{\omega}),$$

where  $\rho, \theta$  are the estimates of  $\rho^*, \theta^*$ , respectively. The design procedure for  $n^* = 2$  suggests using

$$\bar{u}_p = \theta^T \bar{\omega}$$

to obtain  $\hat{e}_1 = \frac{1}{s+p_2} [0]$  and  $e_1 = \frac{1}{s+p_2} \rho^* \tilde{\theta}^T \bar{\omega}$ . However, the suggested control law implies

$$u_p = (s+p_0)(s+p_1)\bar{u}_p = (s+p_0)(s+p_1)\theta^T \bar{\omega}$$

involving  $\ddot{\theta}$ , which is not available for measurement, noting that  $\dot{\theta}$  could be obtained using the adaptive law chosen. Hence the choice of  $\bar{u}_p = \theta^T \bar{\omega}$  is not feasible.

The difficulty of not being able to extend the results for  $n^* = 1, 2$  to  $n^* \geq 3$  was circumvented only after the late 1970s. After that several successful MRAC schemes were proposed using different approaches [2,3,6]. In the following we present a design very similar to that in [3]. We first rewrite the equations for  $e_1$  and  $\hat{e}_1$ :

$$e_1 = \frac{1}{s+p_2} \rho^* (e_0 + \tilde{\theta}^T \bar{\omega}), \quad \hat{e}_1 = \frac{1}{s+p_2} \rho e_0, \quad e_0 = \bar{u}_p - \theta^T \bar{\omega},$$

where  $\tilde{\theta} = \theta - \theta^*$ . We select  $u_p$  so that  $e_0$  goes to zero as  $t \rightarrow \infty$ . In this case, the estimation error  $\varepsilon_1 = e_1 - \hat{e}_1$  satisfies

$$\varepsilon_1 = e_1 - \hat{e}_1 = \frac{1}{s+p_2} (\rho^* \tilde{\theta}^T \bar{\omega} - \tilde{\rho} e_0).$$

Considering the Lyapunov-like function  $V = \frac{\varepsilon_1^2}{2} + \frac{\tilde{\theta}^T \Gamma^{-1} \tilde{\theta}}{2} |\rho^*| + \frac{\hat{\rho}^2}{2\gamma}$ , we design the adaptive law

$$\dot{\theta} = -\Gamma \varepsilon_1 \bar{\omega} \operatorname{sgn}(\rho^*), \quad \dot{\rho} = \gamma \varepsilon_1 e_0, \quad (5C.3)$$

where  $\Gamma = \Gamma^T$  and  $\gamma > 0$ . Let us now express  $e_0$  as

$$\dot{e}_0 = -p_0 e_0 + u_1 - \dot{\theta}^T \bar{\omega} - \theta^T \bar{\omega}_1,$$

where

$$u_1 = \frac{1}{s + p_1} u_p, \quad \bar{\omega}_1 = (s + p_0) \bar{\omega} = \frac{1}{s + p_1} \omega.$$

Substituting for  $\dot{\theta}$ , we obtain

$$\dot{e}_0 = -p_0 e_0 + u_1 + \bar{\omega}^T \Gamma \bar{\omega} \varepsilon_1 \operatorname{sgn}(\rho^*) - \theta^T \bar{\omega}_1.$$

To counteract the term  $\bar{\omega}^T \Gamma \bar{\omega} \varepsilon_1 \operatorname{sgn}(\rho^*)$ , we use a so-called nonlinear damping term [4,5] in  $u_1$  as follows:

$$u_1 = \theta^T \bar{\omega}_1 - \alpha_0 (\bar{\omega}^T \Gamma \bar{\omega})^2 e_0, \quad (5C.4)$$

where  $\alpha_0 > 0$  is a design constant. Using the above control law, we obtain

$$\dot{e}_0 = -\left(p_0 + \alpha_0 (\bar{\omega}^T \Gamma \bar{\omega})^2\right) e_0 + \bar{\omega}^T \Gamma \bar{\omega} \varepsilon_1 \operatorname{sgn}(\rho^*),$$

which will force  $e_0$  and hence  $e_1$  to converge to zero. To check the implementability of the control law, we substitute (5C.4) into  $u_p = (s + p_1)u_1$  and obtain

$$\begin{aligned} u_p &= \theta^T \omega + \dot{\theta}^T \bar{\omega}_1 - (s + p_1) \alpha_0 (\bar{\omega}^T \Gamma \bar{\omega})^2 e_0 = \theta^T \omega + \dot{\theta}^T \bar{\omega}_1 - 4\alpha_0 \bar{\omega}^T \Gamma \bar{\omega} (\bar{\omega}^T \Gamma \bar{\omega}) \dot{e}_0 \\ &\quad - \alpha_0 (p_1 - p_0) (\bar{\omega}^T \Gamma \bar{\omega})^2 e_0 + \alpha_0^2 (\bar{\omega}^T \Gamma \bar{\omega})^4 e_0 - \alpha_0 (\bar{\omega}^T \Gamma \bar{\omega})^3 \varepsilon_1 \operatorname{sgn}(\rho^*), \end{aligned} \quad (5C.5)$$

where  $\dot{\bar{\omega}} = \frac{s}{(s + p_0)(s + p_1)} \omega$ . Hence the control law can be implemented without using differentiators. The stability properties of the above MRAC scheme are summarized in the following theorem.

**Theorem 5.4.6** *The MRAC scheme (5C.3), (5C.5) guarantees that*

- (i) *all signals in the closed-loop plant are bounded and  $e_0(t), e_1(t) \rightarrow 0$  as  $t \rightarrow \infty$ ;*
- (ii) *the estimate  $\rho$  converges to a constant  $\bar{\rho}$  asymptotically independent of the richness of  $r$ . If  $r$  is sufficiently rich of order  $2n$  and  $Z_p, R_p$  are coprime, then the parameter error  $\tilde{\theta} = \theta - \theta^*$  and tracking error  $e_1$  converge to zero*

asymptotically. Further, if  $k_p$  is known, then  $\tilde{\theta}, e_1$  converge to zero exponentially fast.

**Proof [1].** (i) Observe that

$$\begin{aligned}\dot{\varepsilon}_1 &= -p_2\varepsilon_1 + \rho^*\tilde{\theta}^T\bar{\omega} - \tilde{\rho}e_0, \\ \dot{e}_0 &= -(p_0 + \alpha_0(\bar{\omega}^T\Gamma\bar{\omega})^2)e_0 + \bar{\omega}^T\Gamma\bar{\omega}\varepsilon_1 \operatorname{sgn}(\rho^*), \\ \dot{\tilde{\theta}} &= -\Gamma\varepsilon_1\phi \operatorname{sgn}(\rho^*), \\ \dot{\tilde{\rho}} &= \gamma\varepsilon_1e_0.\end{aligned}\tag{5C.6}$$

The time derivative of the Lyapunov-like function

$$V = \frac{\varepsilon_1^2}{2} + |\rho^*|\tilde{\theta}^T \frac{\Gamma^{-1}}{2}\tilde{\theta} + \frac{\tilde{\rho}^2}{2\gamma} + \gamma_0 \frac{e_0^2}{2},$$

where  $\gamma_0 > 0$  is a design constant to be selected, satisfies

$$\begin{aligned}\dot{V} &= -p_2\varepsilon_1^2 - \gamma_0 p_0 e_0^2 - \gamma_0 \alpha_0 e_0^2 (\bar{\omega}^T \Gamma \bar{\omega})^2 + \gamma_0 \varepsilon_1 e_0 \bar{\omega}^T \Gamma \bar{\omega} \operatorname{sgn}(\rho^*) \\ &\leq -p_2\varepsilon_1^2 - \gamma_0 p_0 e_0^2 - \gamma_0 \alpha_0 e_0^2 (\bar{\omega}^T \Gamma \bar{\omega})^2 + \gamma_0 |\varepsilon_1| |e_0| |\bar{\omega}^T \Gamma \bar{\omega}| \\ &\leq -p_2 \frac{\varepsilon_1^2}{2} - \gamma_0 p_2 e_0^2 - \left[ \alpha_0 - \frac{\gamma_0}{2p_2} \right] \gamma_0 e_0^2 (\bar{\omega}^T \Gamma \bar{\omega})^2.\end{aligned}$$

Choosing  $\gamma_0 = 2\alpha_0 p_2$ , we have

$$\dot{V} \leq -p_2 \frac{\varepsilon_1^2}{2} - \gamma_0 p_0 e_0^2 \leq 0.$$

Hence,  $\varepsilon_1, e_0, \rho, \theta \in \mathcal{L}_\infty$  and  $\varepsilon_1, e_0 \in \mathcal{L}_2$ , and therefore  $\hat{e}_1, e_1 \in \mathcal{L}_\infty \cap \mathcal{L}_2; y_p \in \mathcal{L}_\infty$ . This further implies that  $\bar{\omega} = H_0(s)\omega \in \mathcal{L}_\infty$ . (5C.6) together with  $\varepsilon_1, \bar{\omega} \in \mathcal{L}_\infty, \varepsilon_1 \in \mathcal{L}_2$  implies that  $\dot{\theta}, \dot{\rho} \in \mathcal{L}_\infty \cap \mathcal{L}_2$ . (5C.6) further implies that  $\dot{\varepsilon}_1, \dot{y}_p \in \mathcal{L}_\infty$  and hence  $\bar{\omega}, \bar{\omega}_1 \in \mathcal{L}_\infty$ . Since  $\dot{\theta}, \dot{\phi}, \dot{e}_0 \in \mathcal{L}_\infty$ , we have  $\ddot{e}_1 \in \mathcal{L}_\infty$  which together with  $\ddot{y}_p = \ddot{e}_1 + s^2 W_m(s)r$  and properness of  $s^2 W_m(s)$  implies that  $\ddot{y}_p, \ddot{\omega} \in \mathcal{L}_\infty$  and hence  $\omega = (s + p_0)(s + p_1)\bar{\omega} \in \mathcal{L}_\infty$ . Therefore, all the closed-loop signals are bounded. Since  $\dot{e}_0, \dot{e}_1 \in \mathcal{L}_\infty$  and  $e_0, e_1 \in \mathcal{L}_2$ , it follows that  $e_0(t), e_1(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

(ii) Since  $\varepsilon_1, e_0 \in \mathcal{L}_2$  we have  $\int_0^t |\dot{\tilde{\rho}}| d\tau \leq \gamma \int_0^t |\varepsilon_1| |e_0| d\tau \leq \gamma \left( \int_0^\infty \varepsilon_1^2 d\tau \right)^{\frac{1}{2}} \left( \int_0^\infty e_0^2 d\tau \right)^{\frac{1}{2}} \in \mathcal{L}_\infty$  i.e.,  $\dot{\tilde{\rho}} \in \mathcal{L}_1$  and therefore  $\rho, \tilde{\rho}$  converge to a constant as  $t \rightarrow \infty$ .

It can be established that  $\bar{\omega}$  is PE, provided that  $r$  is sufficiently rich of order  $2n$  and  $Z_p, R_p$  are coprime, using similar arguments and steps as in the proof of Theorem 5.4.3.

If  $k_p$  is known, then  $\tilde{\rho} = 0$  and (5C.6) is reduced to

$$\begin{aligned}\dot{\varepsilon}_1 &= -p_2\varepsilon_1 + \rho^*\tilde{\theta}^T\bar{\omega}, \\ \dot{\tilde{\theta}} &= -\Gamma\varepsilon_1\bar{\omega} \operatorname{sgn}(\rho^*).\end{aligned}$$



Since  $\bar{\omega}, \dot{\bar{\omega}} \in \mathcal{L}_\infty$  and  $\bar{\omega}$  is PE, using the same steps as in the proof of Theorem 3.6.1 we can show that  $\tilde{\theta}$  and  $e_1$  converge to zero exponentially fast.

If  $k_p$  is unknown, we consider the equations for  $\dot{\varepsilon}_1, \dot{\tilde{\theta}}$  in (5C.6), i.e.,

$$\begin{aligned}\dot{\varepsilon}_1 &= -p_2 \varepsilon_1 + \rho^* \tilde{\theta}^T \bar{\omega} - \tilde{\rho} e_0, \\ \dot{\tilde{\theta}} &= -\Gamma \varepsilon_1 \bar{\omega} \operatorname{sgn}(\rho^*),\end{aligned}$$

as an LTV system with  $\varepsilon_1, \tilde{\theta}$  as the states and  $\tilde{\rho} e_0$  as the input. Since for the above system (as shown above)  $\tilde{\theta}$  and  $e_1$  converge to zero exponentially fast when  $\tilde{\rho} e_0 = 0$  and  $\bar{\omega}$  is PE, and since (as shown in part (i))  $\tilde{\rho} \in \mathcal{L}_\infty$ ,  $e_0 \in \mathcal{L}_\infty \cap \mathcal{L}_2$ , and  $e_0(t) \rightarrow 0$  as  $t \rightarrow \infty$ , it follows from Lemma A.5.9 (in the Appendix) that  $\varepsilon_1(t), \tilde{\theta}(t) \rightarrow 0$  as  $t \rightarrow \infty$ .  $\square$

## 5.5 Direct MRAC with Normalized Adaptive Laws

**Lemma 5.5.2.** *Consider the plant equation (5.72) and the MRAC scheme (5.116), (5.119). There exists a  $\delta > 0$  such that the fictitious normalizing signal*

$$m_f^2 \triangleq 1 + \|u_p\|^2 + \|y_p\|^2,$$

where  $\|\cdot\|$  denotes the  $\mathcal{L}_{2\delta}$  norm,<sup>4</sup> satisfies the following:

- (i)  $\omega_1/m_f, \omega_2/m_f \in \mathcal{L}_\infty$ .
- (ii) If  $\theta \in \mathcal{L}_\infty$  and  $W(s)$  is a proper transfer function with stable poles, then in addition to (i),  $u_p/m_f, y_p/m_f, \omega/m_f, W(s)\omega/m_f \in \mathcal{L}_\infty$ .
- (iii) If  $\dot{r}, \theta \in \mathcal{L}_\infty$ , then in addition to (i) and (ii),  $\|\dot{y}_p\|/m_f, \|\dot{\omega}\|/m_f \in \mathcal{L}_\infty$ .

**Proof [1].** (i) Since  $\omega_1 = \frac{\alpha(s)}{\Lambda(s)} u_p, \omega_2 = \frac{\alpha(s)}{\Lambda(s)} y_p$ , and the relative degree of each element of  $\frac{\alpha(s)}{\Lambda(s)}$  is at least 1, the result follows from Lemma A.5.9 and the definition of  $m_f$ .

(ii) Since  $\theta \in \mathcal{L}_\infty$ , applying Lemma A.5.9 to (5.120), we obtain

$$|y_p(t)| \leq c + c \|\tilde{\theta}^T \omega\| \leq c + c \|\omega\|,$$

where  $c \geq 0$  denotes any finite constant. Therefore, since

$$\|\omega\| \leq \|\omega_1\| + \|\omega_2\| + \|y_p\| + \|r\| \leq c \|u_p\| + c \|y_p\| + c \leq c m_f + c$$

we have  $y_p/m_f, \omega/m_f \in \mathcal{L}_\infty$ . Since  $u_p = \theta^T \omega$  and  $\theta \in \mathcal{L}_\infty$  this further implies that  $u_p/m_f \in \mathcal{L}_\infty$ . Finally, since  $|W(s)\omega| \leq c \|\omega\|$  for some  $\delta > 0$  such that  $W(s - \delta/2)$  is stable,  $W(s)\omega/m_f \in \mathcal{L}_\infty$ .

<sup>4</sup> The same notation is used in the proof.

(iii) The result follows by differentiating (5.120) and (5.121), using (5.116) and the inequality  $\|\dot{\omega}\| \leq \|\dot{\omega}_1\| + \|\dot{\omega}_2\| + \|\dot{y}_p\| + \|\dot{r}\|$ , and applying Lemma A.5.2.  $\square$

**Proof of Theorem 5.5.1.**

**Step 1. Express the plant input and output in terms of the adaptation error  $\tilde{\theta}^T \omega$**   
This step is already described in detail in the outline of the proof. For  $\theta \in \mathcal{L}_\infty$ , which is guaranteed by the adaptive law, we apply Lemma 5.5.1 to establish that the signal  $m_f$  bounds all signals in the closed-loop adaptive system and some of their derivatives.

**Step 2. Use the swapping lemmas and properties of the  $\mathcal{L}_{2\delta}$  norm to upper bound  $\|\tilde{\theta}^T \omega\|$  with terms that are guaranteed by the adaptive law to have finite  $\mathcal{L}_2$  gains**  
Using Lemma A.11.2, we have

$$\tilde{\theta}^T \omega = F_1(s, \alpha_0) \left( \dot{\tilde{\theta}}^T \omega + \tilde{\theta}^T \dot{\omega} \right) + F(s, \alpha_0) \left( \tilde{\theta}^T \omega \right),$$

where  $F(s, \alpha_0) = \alpha_0^{n^*} (s + \alpha_0)^{-n^*}$ ,  $F_1(s, \alpha_0) = 1 - F(s, \alpha_0)s$ ,  $\alpha_0 > 0$  is an arbitrary constant, and  $n^*$  is the relative degree of  $W_m(s)$ . Similarly, using Lemma A.11.1, we have

$$\tilde{\theta}^T \omega = W_m^{-1}(s) \left( \tilde{\theta}^T W_m(s) \omega + W_c(s) \left( (W_b(s) \omega^T) \dot{\tilde{\theta}} \right) \right),$$

where  $W_b(s), W_c(s)$  are as defined in Lemma A.11.1 for  $W(s) = W_m(s)$ . Combining the two swapping lemma equalities above, we obtain

$$\tilde{\theta}^T \omega = F_1 \left[ \dot{\tilde{\theta}}^T \omega + \tilde{\theta}^T \dot{\omega} \right] + F W_m^{-1} \left[ \tilde{\theta}^T W_m(s) \omega + W_c \left( (W_b \omega^T) \dot{\tilde{\theta}} \right) \right].$$

Expressing the normalized estimation error in (5.119) as  $\varepsilon m_s^2 = -\rho^* \tilde{\theta}^T \phi - \tilde{\rho} \xi$  and substituting in the above equation, we obtain

$$\tilde{\theta}^T \omega = F_1 \left[ \dot{\tilde{\theta}}^T \omega + \tilde{\theta}^T \dot{\omega} \right] + F W_m^{-1} \left[ \frac{1}{\rho^*} \varepsilon m_s^2 + \frac{\tilde{\rho}}{\rho^*} \xi + W_c \left( (W_b \omega^T) \dot{\tilde{\theta}} \right) \right].$$

Using Lemma A.11.1, we have

$$\xi = \theta^T \phi + u_f = -\theta^T W_m(s) \omega + W_m(s) \theta^T \omega = W_c \left[ (W_b \omega^T) \dot{\theta} \right].$$

Hence we can rewrite the expression for  $\tilde{\theta}^T \omega$  as

$$\tilde{\theta}^T \omega = F_1 \left[ \dot{\tilde{\theta}}^T \omega + \tilde{\theta}^T \dot{\omega} \right] + \frac{1}{\rho^*} F W_m^{-1} \left[ \varepsilon m_s^2 + \rho W_c \left( (W_b \omega^T) \dot{\theta} \right) \right].$$

Applying Lemma A.11.2 again, we obtain

$$\|\tilde{\theta}^T \omega\| \leq \frac{c}{\alpha_0} (\|\dot{\theta}^T \omega\| + \|\tilde{\theta}^T \dot{\omega}\|) + c\alpha_0^n (\bar{\omega} + \|\varepsilon m_s^2\|), \quad (5C.7)$$

where  $\bar{\omega} = \rho W_c((W_b \omega^T) \dot{\theta})$ . Using Lemma A.5.2 and Lemma 5.5.2, we have

$$\|\bar{\omega}\| \leq c \|\dot{\theta} m_f\|, \quad \|\dot{\theta}^T \omega\| \leq c \|\dot{\theta} m_f\|, \quad \|\tilde{\theta}^T \dot{\omega}\| \leq c m_f, \quad \|\varepsilon m_s^2\| \leq c \|\varepsilon m_s m_f\|.$$

Therefore, for  $\alpha_0 > 1$  and  $\tilde{g}^2 = \varepsilon^2 m_s^2 + |\dot{\theta}|^2$ , (5C.7) leads to (5.124), where because of  $\varepsilon m_s, \dot{\theta} \in \mathcal{L}_2$ ,  $\tilde{g} \in \mathcal{L}_2$ .

**Step 3. Use the B–G lemma to establish signal boundedness** This step is already described in detail in the outline of the proof.

**Step 4. Show that the tracking error converges to zero** From (5.119) we have

$$e_1 = \varepsilon m_s^2 + \rho \xi.$$

Since  $\xi = \theta^T \phi + W_m[\theta^T \omega] = W_m[\theta^T \omega] - \theta^T W_m[\omega] = W_c[W_b[\omega^T] \dot{\theta}]$ ,  $\varepsilon, \varepsilon m_s, \dot{\theta} \in \mathcal{L}_2$ , and  $\omega, n_s \in \mathcal{L}_\infty$ , we have  $e_1, \xi \in \mathcal{L}_2$ . Furthermore, using (5.118),  $\theta, \dot{\theta} \in \mathcal{L}_\infty$  (as guaranteed by the adaptive law (5.119)), Lemma 5.5.2, and Lemma A.11.1, we have  $\dot{e}_1 \in \mathcal{L}_\infty$ . Therefore, using Lemma A.4.7, we establish that  $e_1(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

**Step 5. Establish that the parameter error converges to zero** By definition, we have

$$\phi = W_m(s) \begin{bmatrix} (sI - F)^{-1} g u_p \\ (sI - F)^{-1} g y_p \\ y_p \\ r \end{bmatrix}.$$

Substituting  $u_p = G_p^{-1}(s)y_p$  and  $y_p = W_m(s)r + e_1$  into this, we have

$$\phi = W_m(s)H_1(s)r + W_m(s)H_2(s)e_1,$$

where

$$H_1(s) = \begin{bmatrix} (sI - F)^{-1} g G_p^{-1}(s) W_m(s) \\ (sI - F)^{-1} g W_m(s) \\ W_m(s) \\ 1 \end{bmatrix}, \quad H_2(s) = \begin{bmatrix} (sI - F)^{-1} g G_p^{-1}(s) \\ (sI - F)^{-1} g \\ 1 \\ 0 \end{bmatrix}.$$

It was established in the proof of Theorem 5.4.1 that if  $r$  is sufficiently rich of order  $2n$ , then  $H_1(s)r$  is PE. Hence applying Lemma 3C.1, since  $W_m(s)$  is stable and minimum phase and  $e_1 \in \mathcal{L}_2$ , it follows that  $\phi$  is PE.

Next, we establish the convergence results. From (5.119), we have

$$\begin{aligned} \varepsilon m_s^2 &= \rho^* \tilde{\theta}^T \phi - \tilde{\rho} \xi, \\ \dot{\theta} &= \Gamma \varepsilon \phi \operatorname{sgn}(k_p/k_m). \end{aligned}$$

Using  $u_p, y_p \in \mathcal{L}_\infty$  we have  $\phi, \dot{\phi} \in \mathcal{L}_\infty$ . Hence, since  $\phi$  is PE and  $\xi \in \mathcal{L}_2$ , applying the arguments in the proof of Theorem 3.9.1, we establish that  $\tilde{\theta}(t), e(t) \rightarrow 0$  as  $t \rightarrow \infty$ .  $\square$

## 5.6.2 Indirect MRAC with Normalized Adaptive Law

### *Proof of Theorem 5.6.1.*

**Step 1. Express the plant input and output in terms of the adaptation error  $\tilde{\theta}^T \omega$**  As in the proof of Theorem 5.5.1 we have

$$\begin{aligned} y_p &= W_m(s) \left( r + \frac{1}{c_0^*} \tilde{\theta}^T \omega \right), \\ u_p &= G_p^{-1}(s) W_m(s) \left( r + \frac{1}{c_0^*} \tilde{\theta}^T \omega \right). \end{aligned}$$

Again as in the proof of Theorem 5.5.1 we use  $\|\cdot\|$  to denote the  $\mathcal{L}_{2\delta}$  norm and define the fictitious signal  $m_f^2 \triangleq 1 + \|u_p\|^2 + \|y_p\|^2$ , for some  $\delta > 0$ , to satisfy Lemma 5.5.2. Following the same steps as in the proof of Theorem 5.5.1, we obtain

$$m_f^2 \leq c + c \|\tilde{\theta}^T \omega\|^2.$$

**Step 2. Use the swapping lemmas and properties of the  $\mathcal{L}_{2\delta}$  norm to upper bound**

**$\|\tilde{\theta}^T \omega\|$  with terms that are guaranteed by the adaptive law to have finite  $\mathcal{L}_2$  gains** Note first that based on (5.133) and (5.134) it can be established using similar arguments as in the proof of Theorem 5.5.1 that  $\theta, 1/\hat{k}_p, \theta_p \in \mathcal{L}_\infty$ ;  $\dot{\theta}, \dot{\theta}_p \in \mathcal{L}_2$ . Using (5.132), we have

$$\begin{aligned} \theta_1^{*T} W_m(s) \omega_1 + \theta_2^{*T} W_m(s) \omega_2 + \theta_3^{*T} W_m(s) y_p &= W_m(s) u_p - \frac{1}{k_p} \bar{Z}_p(s) Q(s) \frac{W_m(s)}{\Lambda(s)} u_p \\ &\quad + \frac{1}{k_p} \left( Q(s) R_p(s) - \Lambda_0(s) R_m(s) \right) \frac{W_m(s)}{\Lambda(s)} y_p. \end{aligned}$$

Similarly, using (5.133), we obtain

$$\begin{aligned} \theta_1^T W_m(s) \omega_1 + \theta_2^T W_m(s) \omega_2 + \theta_3^T W_m(s) y_p &= W_m(s) u_p - \frac{1}{\hat{k}_p} \hat{Z}_p(s, t) \cdot \hat{Q}(s, t) \frac{W_m(s)}{\Lambda(s)} u_p \\ &\quad + \frac{1}{\hat{k}_p} \left( \hat{Q}(s, t) \cdot \hat{R}_p(s, t) - \Lambda_0(s) R_m(s) \right) \frac{W_m(s)}{\Lambda(s)} y_p. \end{aligned}$$

Taking the difference of the two equations above, we get

$$\tilde{\theta}_1^T W_m(s) \omega_1 + \tilde{\theta}_2^T W_m(s) \omega_2 + \tilde{\theta}_3^T W_m(s) y_p = \bar{e}_1 + \bar{e}_2 + \bar{e}_3,$$

where  $\tilde{\theta}_i \triangleq \theta_i - \theta_i^*$  and

$$\begin{aligned}\bar{e}_1 &= \frac{1}{\hat{k}_p} \left( \hat{Q}(s,t) \bullet \hat{R}_p(s,t) \frac{W_m(s)}{\Lambda(s)} y_p - \hat{Q}(s,t) \bullet \hat{Z}_p(s,t) \frac{W_m(s)}{\Lambda(s)} u_p \right), \\ \bar{e}_2 &= \left( \frac{1}{k_p} - \frac{1}{\hat{k}_p} \right) \frac{\Lambda_0(s) R_m(s)}{\Lambda(s)} W_m(s) y_p = (c_0^* - c_0) y_p, \\ \bar{e}_3 &= \frac{Q(s)}{k_p \Lambda(s)} W_m(s) (\bar{Z}_p(s) u_p - R_p(s) y_p) = 0.\end{aligned}$$

Therefore, for  $\bar{\omega} \triangleq [\omega_1^T, \omega_2^T, y_p, W_m^{-1}(s) y_p]^T$  and  $\tilde{\theta} \triangleq \theta - \theta^*$  we have

$$\tilde{\theta}^T W_m(s) \bar{\omega} = \bar{e}_1.$$

Furthermore, applying Lemma A.11.3 to  $\bar{e}_1 = \frac{1}{\hat{k}_p} (\hat{Q}(s,t) \bullet \hat{R}_p(s,t) \frac{W_m(s)}{\Lambda(s)} y_p - \hat{Q}(s,t) \bullet \hat{Z}_p(s,t) \frac{W_m(s)}{\Lambda(s)} u_p)$  we obtain

$$\bar{e}_1 = \bar{e}_{11} - \bar{e}_{12},$$

where

$$\begin{aligned}\bar{e}_{11} &= \frac{1}{\hat{k}_p} \hat{Q}(s,t) \left( \hat{R}_p(s,t) \frac{W_m(s)}{\Lambda(s)} y_p - \hat{Z}_p(s,t) \frac{W_m(s)}{\Lambda(s)} u_p \right), \\ \bar{e}_{12} &= q^T D_{n^*-2}(s) \left[ \alpha_{n^*-2}^T(s) \left[ \alpha_m^T(s) \frac{W_m(s)}{\Lambda(s)} u_p, \alpha_{n-1}^T(s) \frac{W_m(s)}{\Lambda(s)} y_p \right] \right] \dot{\theta}_p, \\ \hat{Q}(s,t) &= q^T \alpha_{n^*-1}(s).\end{aligned}$$

Since

$$\begin{aligned}\bar{e}_{11} &= \frac{1}{\hat{k}_p} \hat{Q}(s,t) \left[ \hat{R}_p(s,t) \frac{W_m(s)}{\Lambda(s)} y_p - \hat{Z}_p(s,t) \frac{W_m(s)}{\Lambda(s)} u_p \right] \\ &= \frac{1}{\hat{k}_p} \hat{Q}(s,t) \left[ -\theta_p^T \frac{W_m(s) \Lambda_p(s)}{\Lambda(s)} \phi \right] = \frac{1}{\hat{k}_p} \hat{Q}(s,t) \left[ -\tilde{\theta}_p^T \frac{W_m(s) \Lambda_p(s)}{\Lambda(s)} \phi \right] \\ &= \frac{1}{\hat{k}_p} \hat{Q}(s,t) \left[ \frac{W_m(s) \Lambda_p(s)}{\Lambda(s)} [\varepsilon m_s^2] - W_c(s) [W_b(s) \phi^T] \dot{\theta}_p \right],\end{aligned}$$

where Lemma A.11.1 is used to obtain the last equality, using Lemma 5.5.2, the definition of  $\hat{Q}(s,t)$ , and the fact that  $\theta \in \mathcal{L}_\infty$  and  $y_p = W_m(s)(r + \frac{1}{c_0^*} \tilde{\theta}^T \omega)$ ,  $u_p = G_p^{-1}(s) W_m(s)(r + \frac{1}{c_0^*} \tilde{\theta}^T \omega)$ , we obtain

$$\|\bar{e}_1\| \leq \|\bar{e}_{11}\| + \|\bar{e}_{12}\| \leq c \|\varepsilon m_s m_f\| + c \|\dot{\theta} m_f\|, \quad (5C.8)$$

where  $c$  denotes a generic positive constant. Furthermore, from (5.117) we have

$$y_p - y_m = W_m \left[ \frac{1}{c_0^*} (u_p - \theta^{*T} \omega) \right] = W_m \left[ \frac{1}{c_0^*} \tilde{\theta}^T \omega \right].$$

Therefore,

$$\tilde{\theta}^T \bar{\omega} - \tilde{\theta}^T \omega = \tilde{\theta}^T (\bar{\omega} - \omega) = (c_0 - c_0^*) (W_m^{-1} y_p - r) = \frac{c_0 - c_0^*}{c_0^*} \tilde{\theta}^T \omega,$$

i.e.,  $\tilde{\theta}^T \bar{\omega} = \frac{c_0}{c_0^*} \tilde{\theta}^T \omega$ . Hence, using Lemma A.11.2, we obtain

$$\tilde{\theta}^T \omega = \frac{c_0^*}{c_0} \left\{ F_1(s, \alpha_0) [\dot{\theta}^T \bar{\omega} + \tilde{\theta}^T \dot{\bar{\omega}}] + F(s, \alpha_0) [\tilde{\theta}^T \bar{\omega}] \right\}, \quad (5C.9)$$

where  $F(s, \alpha_0) = \alpha_0^{n^*} (s + \alpha_0)^{n^*}$ ,  $F_1(s, \alpha_0) = 1 - F(s, \alpha_0)s$ ,  $\alpha_0 > 0$  is an arbitrary constant, and  $n^*$  is the relative degree of  $W_m(s)$  and satisfies  $\|F_1(s, \alpha_0)\|_{\infty \delta} \leq \frac{c}{\alpha_0}$ ,  $\|F(s, \alpha_0)W_m^{-1}(s)\|_{\infty \delta} \leq c\alpha_0^{n^*}$  for any  $\alpha_0 > \delta > 0$ . Applying Lemma A.11.1, we obtain

$$\tilde{\theta}^T \bar{\omega} = W_m^{-1} \left[ \tilde{\theta}^T W_m [\omega_p] + W_c \left[ W_b [\bar{\omega}^T] \dot{\theta} \right] \right] = W_m^{-1} \left[ \bar{e}_1 + W_c \left[ W_b [\bar{\omega}^T] \dot{\theta} \right] \right]. \quad (5C.10)$$

Substituting in (5C.9), we get

$$\tilde{\theta}^T \omega = \frac{c_0^*}{c_0} \left( F_1(s, \alpha_0) [\dot{\theta}^T \bar{\omega} + \tilde{\theta}^T \dot{\bar{\omega}}] + F(s, \alpha_0) W_m^{-1}(s) \left[ \bar{e}_1 + W_c(s) \left[ W_b(s) \bar{\omega}^T \right] \dot{\theta} \right] \right).$$

Since  $c_0$  is bounded from below, i.e., applying Lemma A.5.10 and Lemma 5.5.2, we obtain

$$\|\tilde{\theta}^T \omega\| \leq \frac{c}{\alpha_0} \|\dot{\theta} m_f\| + \frac{c}{\alpha_0} m_f + c\alpha_0^{n^*} (\|\bar{e}_1\| + \|\dot{\theta} m_f\|). \quad (5C.11)$$

Using (5C.8), (5C.11) together with  $\theta, 1/\hat{k}_p, \theta_p \in \mathcal{L}_\infty$ ;  $\dot{\theta}, \dot{\theta}_p \in \mathcal{L}_2$ , we obtain

$$\|\tilde{\theta}^T \omega\| \leq \frac{c}{\alpha_0} m_f + c\alpha_0^{n^*} \|\tilde{g} m_f\|,$$

where  $\tilde{g} \triangleq (\frac{1}{\alpha_0^2} |\dot{\theta}|^2 + \alpha_0^{2n^*} (|\dot{\theta}|^2 + \varepsilon^2 m_s^2 + |\dot{\theta}_p|^2))^{1/2} \in \mathcal{L}_2$ .

**Step 3. Use the B–G lemma to establish signal boundedness** This step is the same as Step 3 of (the outline of) the proof of Theorem 5.5.1.

**Step 4. Show that the tracking error converges to zero** Using  $\tilde{\theta}^T \bar{\omega} = \frac{c_0}{c_0^*} \tilde{\theta}^T \omega$  in (5.117), we have

$$e_1 = y_p - y_m = W_m \left[ \frac{1}{c_0^*} (u_p - \theta^{*T} \omega) \right] = W_m \left[ \frac{1}{c_0^*} \tilde{\theta}^T \omega \right] = W_m \left[ \frac{1}{c_0} \tilde{\theta}^T \bar{\omega} \right].$$

Applying Lemma A.11.1 and substituting (5C.10), we obtain

$$e_1 = \frac{1}{c_0} \left( \bar{e}_1 + W_c(s) [W_b(s) [\bar{\omega}^T] \theta] \right) - W_c(s) \left[ \frac{\dot{c}_0}{c_0^2} W_b(s) [\bar{\omega}^T \tilde{\theta}] \right].$$

Using boundedness of all the signals and parameter estimates, (5C.8), and  $\varepsilon m_s, \dot{\theta}, \dot{\theta}_p \in \mathcal{L}_2$ , this implies that  $e_1 \in \mathcal{L}_2$ . Using similar arguments as in the proof of Theorem 5.5.1, we have  $\dot{e}_1 \in \mathcal{L}_\infty$ . Therefore, using Lemma A.4.7, we establish that  $e_1(t) \rightarrow 0$  as  $t \rightarrow \infty$ .  $\square$

## 5.7.2 Robust Direct MRAC

**Lemma 5.7.8.** Consider the plant equation (5.166) and the MRAC scheme (5.168), (5.169). For any  $\delta_0 > 0$  and any  $\delta \in (0, \delta_0]$ , the fictitious normalizing signal

$$m_f^2 \triangleq 1 + \|u_p\|^2 + \|y_p\|^2,$$

where  $\|\cdot\| \triangleq \|(\cdot)_t\|_{2\delta}$  denotes the  $\mathcal{L}_{2\delta}$  norm,<sup>5</sup> satisfies the following:

- (i)  $\omega_1/m_f, \omega_2/m_f, \|\omega\|/m_f, m_s/m_f \in \mathcal{L}_\infty$ .
- (ii) If  $\theta \in \mathcal{L}_\infty$  and  $W(s)$  is a proper transfer function with all its poles in  $\text{Re}[s] < -\delta_0/2$ , then in addition to (i),  $u_p/m_f, y_p/m_f, \omega/m_f, W(s)\omega/m_f, \|u_p\|/m_f, \|\dot{y}_p\|/m_f \in \mathcal{L}_\infty$ .
- (iii) If  $\dot{r}, \theta \in \mathcal{L}_\infty$ , then in addition to (i) and (ii),  $\|\dot{\omega}\|/m_f \in \mathcal{L}_\infty$ .
- (iv) For  $\delta = \delta_0$ ,  $m_f = m_s$ .

*Proof.* (i) Using the same arguments as in the proof of Lemma 5.5.2(i), we have  $\omega_1/m_f, \omega_2/m_f, \|\omega\|/m_f \in \mathcal{L}_\infty$ . Using  $m_s^2 = 1 + \|u_{pr}\|_{2\delta_0}^2 + \|y_{pr}\|_{2\delta_0}^2 \leq 1 + \|u_{pr}\|_{2\delta}^2 + \|y_{pr}\|_{2\delta}^2$ , we further have  $m_s/m_f \in \mathcal{L}_\infty$ .

- (ii) Rewriting the equation for  $e_1 = y_p - y_m$  right before (5.169) as

$$y_p = \rho^* W_m(s) \tilde{\theta}^T \omega + \rho^* \eta + \rho^* d + W_m(s) r,$$

and using the expressions for  $\eta, d$ , the fact that  $W_m, W_m \Delta_m$  are strictly proper with all their poles in  $\text{Re}[s] < -\delta_0/2$ , and  $\theta, \tilde{\theta} \in \mathcal{L}_\infty$ , we establish that

$$|y_p(t)| \leq c \|\omega\| + c \|u_p\| + c \leq c m_f + c;$$

i.e.,  $y_p/m_f \in \mathcal{L}_\infty$ . Using (i), this further implies that  $\omega/m_f \in \mathcal{L}_\infty$ . Hence, since  $u_p = \theta^T \omega$  and  $\theta, \|\omega\|/m_f \in \mathcal{L}_\infty$ , we also have  $\frac{u_p}{m_f}, \frac{\|u_p\|}{m_f} \in \mathcal{L}_\infty$ . Using Lemma A.5.2, we further have  $W(s)\omega/m_f$ . Finally,  $\|\dot{y}_p\|/m_f \in \mathcal{L}_\infty$  follows by taking the derivative of both sides of

$$y_p = \rho^* W_m(s) \tilde{\theta}^T \omega + \rho^* \eta + \rho^* d + W_m(s) r,$$

<sup>5</sup> The same notation is used in the proofs below.

observing that  $\|\dot{\eta} + \dot{d}\| \leq \left\| sW_m(s) \frac{\Lambda - \theta_1^{*T} \alpha(s)}{\Lambda(s)} \Delta_m(s) \right\|_{\infty \delta} \|u_p\| + cd_0 \leq cm_f + cd_0$  and that  $sW_m(s)$  is proper.

(iii) The result is established using the same arguments as in the proof of Lemma 5.5.2(iii).

(iv) The result follows by definition.  $\square$

**Proof of Theorem 5.7.5.**

**Step 1. Express the plant input and output in terms of the parameter error term  $\tilde{\theta}^T \omega$ .** From (5.137), (5.138), we have

$$y_p = \frac{G_0 \Lambda c_0^*}{\Lambda - \theta_1^{*T} \alpha - G_0 (\theta_2^{*T} \alpha + \theta_3^* \Lambda)} \left[ r + \frac{1}{c_0^*} \tilde{\theta}^T \omega + \frac{\Lambda - \theta_1^{*T} \alpha}{c_0^* \Lambda} \bar{\eta} \right],$$

$$u_p = \frac{\Lambda c_0^*}{\Lambda - \theta_1^{*T} \alpha - G_0 (\theta_2^{*T} \alpha + \theta_3^* \Lambda)} \left[ r + \frac{1}{c_0^*} \tilde{\theta}^T \omega \right] + \frac{G_0 D_1^*}{\Lambda - \theta_1^{*T} \alpha - G_0 (\theta_2^{*T} \alpha + \theta_3^* \Lambda)} \bar{\eta},$$

where  $\bar{\eta} \triangleq \Delta_m u_p + (\Delta_m + 1) d_u$ . Substituting (5.139), we obtain

$$y_p = W_m \left( r + \frac{1}{c_0^*} \tilde{\theta}^T \omega \right) + \eta_y,$$

$$u_p = G_0^{-1} W_m \left( r + \frac{1}{c_0^*} \tilde{\theta}^T \omega \right) + \eta_u,$$

where  $\eta_u \triangleq \frac{\theta_2^{*T} \alpha + \theta_3^* \Lambda}{c_0^* \Lambda} W_m \bar{\eta}$ ,  $\eta_y \triangleq \frac{\Lambda - \theta_1^{*T} \alpha}{c_0^* \Lambda} W_m \bar{\eta}$ . Since  $W_m, G_0^{-1} W_m$  are stable, applying Lemma A.5.2, this implies

$$\|y_p\| \leq c + c \|\tilde{\theta}^T \omega\| + \left\| \frac{\Lambda - \theta_1^{*T} \alpha}{c_0^* \Lambda} \right\|_{\infty \delta} \|W_m \Delta_m\|_{\infty \delta} \|u_p\| + cd_0 \leq c + c \|\tilde{\theta}^T \omega\| + c \|W_m \Delta_m\|_{\infty \delta} m_f,$$

$$\|u_p\| \leq c + c \|\tilde{\theta}^T \omega\| + \left\| \frac{\theta_2^{*T} \alpha + \theta_3^* \Lambda}{c_0^* \Lambda} \right\|_{\infty \delta} \|W_m \Delta_m\|_{\infty \delta} \|u_p\| + cd_0 \leq c + c \|\tilde{\theta}^T \omega\| + c \|W_m \Delta_m\|_{\infty \delta} m_f,$$

i.e.,  $m_f^2 \leq c + c \|\tilde{\theta}^T \omega\|^2 + c \|W_m \Delta_m\|_{\infty \delta}^2 m_f^2$ , where  $c$  denotes a generic positive constant.

**Step 2. Use the swapping lemmas and properties of the  $\mathcal{L}_{2\delta}$  norm to bound  $\|\tilde{\theta}^T \omega\|$  from above with terms that are guaranteed by the robust adaptive laws to have small gains in the m.s.s.** Using the same notation and the same steps as in the proof of Theorem 5.5.1, we have

$$\tilde{\theta}^T \omega = F_1 \left[ \dot{\tilde{\theta}}^T \omega + \tilde{\theta}^T \dot{\omega} \right] + F W_m^{-1} \left[ \tilde{\theta}^T W_m \omega + W_c \left( (W_b \omega^T) \dot{\tilde{\theta}} \right) \right]. \quad (5C.12)$$

Using (5.169), we get



$$\varepsilon m_s^2 = \rho^* (\theta^{*T} \phi + W_m \theta^T \omega + \eta + d) - \rho (\theta^T \phi + W_m \theta^T \omega) = \rho^* (\tilde{\theta}^T W_m \omega + \eta + d) - \tilde{\rho} W_c \left( (W_b \omega^T) \dot{\tilde{\theta}} \right)$$

and hence

$$\tilde{\theta}^T W_m \omega = \frac{\varepsilon m_s^2}{\rho^*} + \frac{\tilde{\rho}}{\rho^*} W_c \left( (W_b \omega^T) \dot{\tilde{\theta}} \right) - \eta - d.$$

Substituting in (5C.12), we obtain

$$\tilde{\theta}^T \omega = F_1 \left[ \dot{\tilde{\theta}}^T \omega + \tilde{\theta}^T \dot{\omega} \right] + \frac{F W_m^{-1}}{\rho^*} \left[ \varepsilon m_s^2 + \rho W_c \left( (W_b \omega^T) \dot{\tilde{\theta}} \right) \right] - F W_m^{-1} [\eta + d].$$

Following the same steps as in the proof of Theorem 5.5.1, we obtain

$$\begin{aligned} \|\tilde{\theta}^T \omega\| &\leq \frac{c}{\alpha_0} \left( \|\dot{\tilde{\theta}}^T \omega\| + \|\tilde{\theta}^T \dot{\omega}\| \right) + c \alpha_0^k \left( \|\varepsilon m_s^2\| + \left\| W_c \left( (W_b \omega^T) \dot{\tilde{\theta}} \right) \right\| + \|\eta + d\| \right) \\ &\leq c \|g m_f\| + \frac{c}{\alpha_0} m_f + c \alpha_0^k \Delta_\infty + c d_0, \end{aligned}$$

where

$$g = \sqrt{\frac{|\dot{\theta}|^2}{\alpha_0^2} + \alpha_0^{2k} \left( \varepsilon^2 m_s^2 + |\dot{\theta}|^2 \right)} \in \mathcal{S} \left( \Delta_2^2 + \frac{d_0^2}{m_s^2} \right)$$

since  $\varepsilon m_s, |\dot{\theta}| \in \mathcal{S}(\eta^2/m_s^2 + d_0^2/m_s^2)$  and  $|\eta| \leq \Delta_2 m_s$ .

**Step 3. Use the B–G lemma to establish boundedness** This step is explained in detail in the outline of the proof.

**Step 4. Establish bounds for the tracking error  $e_1$ .** It follows from (5.169) that

$$e_1 = \varepsilon m_s^2 + \rho \xi.$$

Since  $\varepsilon, \varepsilon m_s, \rho \in \mathcal{S}(\eta^2/m_s^2 + d^2/m_s^2)$  and  $\xi, m_s \in \mathcal{L}_\infty$ , it follows that  $e_1 \in \mathcal{S}(\eta^2/m_s^2 + d^2/m_s^2)$ . By definition of  $\Delta$ , this implies that  $e_1 \in \mathcal{S}(\Delta^2 + d_0^2)$ , i.e.,

$$\frac{1}{T} \int_t^{t+T} e_1^2 d\tau \leq c(\Delta^2 + d_0^2) + \frac{c}{T} \quad \forall t \geq 0, T > 0.$$

**Step 5. Establish convergence of the estimated parameter and tracking errors to residual sets** From the definition of  $\phi$  we have

$$\phi = W_m(s) \left[ \frac{\alpha^T(s)}{\Lambda(s)} u_p, \frac{\alpha^T(s)}{\Lambda(s)} y_p, y_p, r \right]^T.$$

Using  $y_p = W_m(s)r + e_1$  and (5.166), we have

$$u_p = G_0^{-1}(s)y_p - \eta_y = G_0^{-1}(s)(W_m(s)r + e_1) - \bar{\eta}_y,$$

where  $\bar{\eta}_y = \Delta_m(s)(u_p + d_u) + d_u$ , and hence

$$\phi = \phi_m + \phi_e,$$

where

$$\begin{aligned}\phi_m &= W_m(s) \left[ G_0^{-1}(s)W_m(s) \frac{\alpha^T(s)}{\Lambda(s)}, W_m(s) \frac{\alpha^T(s)}{\Lambda(s)}, W_m(s), 1 \right]^T [r], \\ \phi_e &= W_m(s) \left[ G_0^{-1}(s) \frac{\alpha^T(s)}{\Lambda(s)}, \frac{\alpha^T(s)}{\Lambda(s)}, 1, 0 \right]^T [e_1] - W_m(s) \left[ \frac{\alpha^T(s)}{\Lambda(s)}, 0_{n+1}^T \right]^T [\bar{\eta}_y].\end{aligned}$$

Lemma 3.11.6 implies that  $\phi_m$  is PE with level  $\alpha_0 > O(\Delta^2 + d_0^2)$  if  $r$  is dominantly rich and  $Z_p, R_p$  are coprime. Hence there exist  $T > 0$  such that

$$\frac{1}{T} \int_t^{t+T} \phi_m(\tau) \phi_m^T(\tau) d\tau \geq \alpha_0 I \quad \forall t \geq 0.$$

Furthermore, since  $e_1, \bar{\eta}_y \in \mathcal{S}(\Delta^2 + d_0^2)$  and  $u_p \in \mathcal{L}_\infty$ , we have  $\phi_e \in \mathcal{S}(\Delta^2 + d_0^2)$  and hence

$$\begin{aligned}\frac{1}{lT} \int_t^{t+lT} \phi(\tau) \phi^T(\tau) d\tau &\geq \frac{1}{2lT} \int_t^{t+lT} \phi_m(\tau) \phi_m^T(\tau) d\tau - \frac{1}{lT} \int_t^{t+lT} \phi_e(\tau) \phi_e^T(\tau) d\tau \\ &\geq \left( \frac{\alpha_0}{2} - c(\Delta^2 + d_0^2) - \frac{c}{lT} \right) I\end{aligned}$$

for any positive integer  $l$ . Therefore, for any  $l$  satisfying  $\frac{c}{lT} < \frac{\alpha_0}{8}$  and for  $c(\Delta^2 + d_0^2) < \frac{\alpha_0}{8}$ , we have

$$\frac{1}{lT} \int_t^{t+lT} \phi(\tau) \phi^T(\tau) d\tau \geq \frac{\alpha_0}{4} I,$$

which implies that  $\phi$  is PE.

Applying the arguments in the proof of Theorem 3.12.2, we establish that when  $\phi$  is PE, (5.169) guarantees that  $\tilde{\theta}$  converges to a residual set whose size is of the order of the modeling error; i.e.,  $\tilde{\theta}$  satisfies  $|\tilde{\theta}(t)| \leq c(\Delta + d_0) + r_{\tilde{\theta}}(t)$ , where  $\lim_{t \rightarrow \infty} r_{\tilde{\theta}}(t) = 0$  and the convergence is exponentially fast. From (5.171), we have

$$e_1(t) = \frac{1}{c_0^*} W_m(s) [\tilde{\theta}^T \omega] + \eta_y.$$

Therefore, since  $\omega \in \mathcal{L}_\infty$  and  $|\eta_y| \leq c(\Delta_2 + d_0)$ , we obtain

$$|e_1(t)| \leq c(\Delta + d_0) + cr_e(t),$$

where  $r_e \triangleq W_m[r_{\tilde{\theta}}]$ . Exponential convergence of  $r_{\tilde{\theta}}(t)$  to zero implies that  $r_e(t) \rightarrow 0$  exponentially as  $t \rightarrow \infty$  as well. Thus, we have established that  $e_1, \tilde{\theta}$  converges exponentially to the residual set  $S$  defined in the theorem.  $\square$

## 5.10 Examples Using the Adaptive Control Toolbox

The MRC and MRAC algorithms presented in this chapter can be implemented using a set of MATLAB functions and the Simulink block **Adaptive Controller** provided in the Adaptive Control Toolbox. For implementation of MRAC algorithms these tools need to be used together with the PI functions or blocks introduced in section 3.16. In this section, we demonstrate use of the Adaptive Control Toolbox in various MRC and MRAC problems via a number of simulation examples.

### 5.10.1 MRC

In section 5.3 we have seen that for a given plant of the form (5.73) and a reference model of the form (5.76), if the plant parameters are known, then an MRC scheme can be constructed to force the closed-loop system (the integration of the plant (5.73) and the MRC) to behave as the reference model. The control parameters of this MRC scheme are obtained by solving (5.84). The MATLAB function **mrcpoly** can be used to solve (5.84) for a given design polynomial  $\Lambda_0(s)$ . Denoting the coefficient vectors of  $Z(s), R(s), Z_m(s), R_m(s), \Lambda_0(s)$  as  $Z, R, Z_m, R_m, L0$ , respectively,

```
[thetau, thetay, thetar, RETYPE] = mrcpoly(Z, R, Zm, Rm, L0)
```

returns the parameter vectors  $\theta_u^*, \theta_y^*, \theta_r^*$ . RETYPE is returned as 0 if the equation solving process is successful and as -1 if it fails.

**Example 5.10.1** Consider the plant  $y = \frac{1}{s(s+3)^2}u$ . It is necessary to choose  $u$  so that  $y$  tracks the reference signal  $y_m$  of the reference model  $y_m = \frac{1}{(s+1)^3}r$  with  $r(t) = 1 + \sin(4t + \frac{\pi}{6})$ . The control signal for this task can be generated as

$$u = \theta_u^{*T} \frac{\alpha(s)}{\Lambda(s)} u + \theta_y^{*T} \begin{bmatrix} \frac{\alpha(s)}{\Lambda(s)} y \\ y \end{bmatrix} + \theta_r^* r, \quad \alpha(s) = [s, 1]^T,$$

choosing the degree of  $\Lambda(s) = \Lambda_0(s)$  as  $n-1=2$ . Fixing  $\Lambda(s) = \Lambda_0(s) = s^2 + s + 1$ ,  $\theta_u^*, \theta_y^*, \theta_r^*$  can be determined using **mrcpoly**:

```
Zp = 1;
Rp = [1 6 9 0];
Zm = 1;
Rm = [1 3 3 1];
Lambda0 = [1 1 1];
[thetau, thetay, thetar, RETYPE] =
mrcpoly(Zp, Rp, Zm, Rm, Lambda0)
```

The resultant parameter vectors are  $\theta_u^* = [3 \ -9]^T$ ,  $\theta_y^* = [51 \ -36 \ 35]$ ,  $\theta_r^* = 1$ , and the resultant control law is

$$u = \frac{3s-9}{s^2+s+1}u + \frac{51s-36}{s^2+s+1}y + 35y + r. \quad \blacksquare$$

## 5.10.2 Direct MRAC

When the plant parameters are unknown, we can no longer solve (5.84) to obtain the controller parameters. Instead, we need to use an adaptive control scheme which produces the estimates of the control parameters directly based on an appropriate parametric model involving these control parameters (direct adaptive control), or which first produces the estimates of the plant parameters based on another appropriate parametric model and then calculates the control parameters using these plant parameter estimates (indirect). A set of direct MRAC schemes have been introduced in sections 5.4 and 5.5, and some indirect MRAC schemes in section 5.6.

The direct MRAC schemes with normalized adaptive laws of section 5.5 can be implemented using the MATLAB functions **umrcdr1**, **umrcdrb** or the Simulink block **Adaptive Controller** of the Adaptive Control Toolbox, incorporated with the parameter identification functions (or the Simulink block **Parameter Estimator**) described in Chapter 3. **umrcdr1** is used for schemes based on SPM or DPM, while **umrcdrb** is for B-SPM and B-DPM. Examples 5.10.2–5.10.7 demonstrate various uses of these tools.

**Example 5.10.2** Consider the plant and the reference model of Example 5.10.1. The corresponding linear SPM can be written as

$$z = \theta^{*T} \phi$$

where

$$\begin{aligned} \theta^* &= [\theta_u^{*T}, \theta_y^{*T}, \theta_r^*]^T, \\ z &= W_m(s)u \\ \phi &= \left[ W_m(s)(H(s)u)^T, W_m(s)(H(s)y)^T, W_m(s)y, y \right]^T, \\ W_m(s) &= \frac{1}{(s+1)^3}, \quad H(s) = \frac{\alpha(s)}{\Lambda(s)} = \frac{[s, 1]^T}{s^2+s+1}. \end{aligned}$$

In Example 5.10.1, we have calculated the parameter vectors as  $\theta_u^* = [3 \ -9]^T$ ,  $\theta_y^* = [51 \ -36 \ 35]$ ,  $\theta_r^* = 1$ , and the control law as  $u = \frac{3s-9}{s^2+s+1}u + \frac{51s-36}{s^2+s+1}y + 35y + r$ . Now, we can generate the control signal as well as the signals in the SPM using the following code (added to the code of Example 5.10.1):

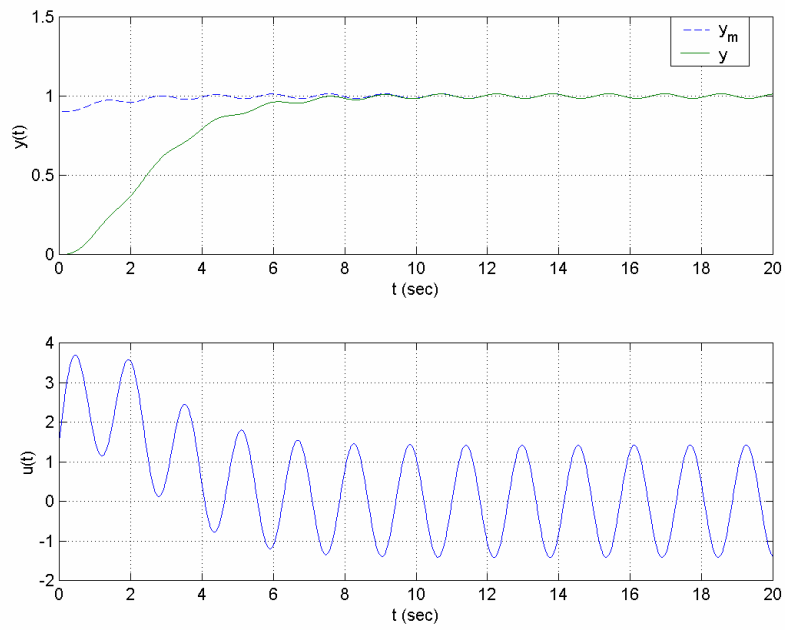
```
Lambda = conv(Zm, Lambda0);
theta = [thetau(:); thetay(:); thetar(:)];
dt = 0.01; % Time increment for simulation (sec).
t = [0:dt:20]; % Process time (sec).
tprev = [-5:dt:-dt];
ltprev = length(tprev);
```

```

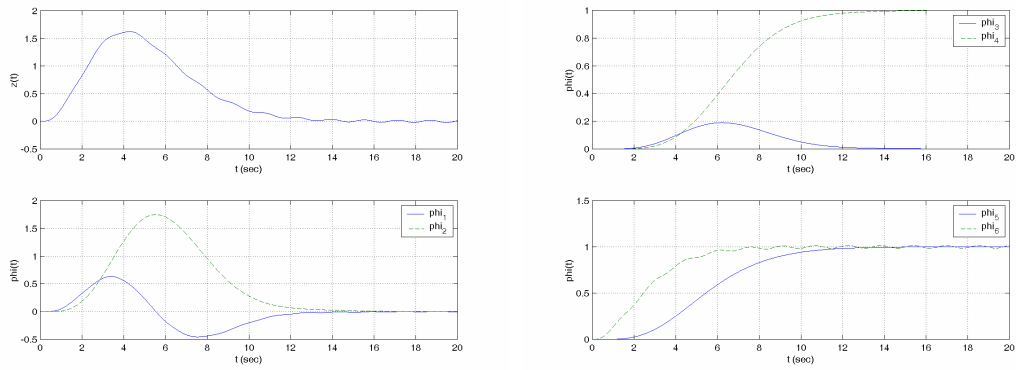
lt = length(t);
r = sin(4*t + pi/6) + 1;           % Reference signal.
rprev = sin(4*tprev + pi/6) + 1;
% State initialization for the plant:
[nstate, xp0] = ufilt('init',Zp,Rp,0);
xp(:,1) = xp0;
% State initialization for the reference model:
[nstate, xm0] = ufilt('init',Zm,Rm,0);
xm(:,1) = xm0;
for k = 1:ltprev,
    dxm = ufilt('state',xm(:,1),rprev(k),Zm,Rm);
    xm(:,1)=xm(:,1) + dt*dxm;
end
% State initialization for linear parametric model:
[nstate xl0] = umrcdr1('init',[3 0],Lambda,Zm,Rm);
xl(:,1) = xl0;
% Signal initialization:
ym(1) = ufilt('output',xm(:,1),Zm,Rm);
y(1) = 0;
% Process:
for k = 1:lt,
    u(k) = umrcdr1('control',xl(:,k),[y(k) r(k)],[3
0],Lambda,Zm,Rm,theta);
    [z(k), phi(:,k)] = umrcdr1('output',xl(:,k),[u(k)
y(k)],[3 0],Lambda,Zm,Rm);
    dxl = umrcdr1('state',xl(:,k),[u(k) y(k)],Lambda,Zm,Rm);
    xl(:,k+1)=xl(:,k) + dt*dxl;
    dxm = ufilt('state',xm(:,k),r(k),Zm,Rm);
    xm(:,k+1)=xm(:,k) + dt*dxm;
    ym(k+1) = ufilt('output',xm(:,k+1),Zm,Rm);
    dxp = ufilt('state',xp(:,k),u(k),Zp,Rp);
    xp(:,k+1)=xp(:,k) + dt*dxp;
    y(k+1) = ufilt('output',xp(:,k+1),Zp,Rp);
end
% Outputs:
ym = ym(1:lt);
y = y(1:lt);

```

The results are shown in Figures 5C.1 and 5C.2. Another means of simulation is using the Simulink block **Adaptive Controller**. One can use the scheme shown in Figure 5C.3, and choose the appropriate options and enter appropriate parameters in the menus of **Adaptive Controller**. ■



**Figure 5C.1** Output and control signals of Example 5.10.2.



**Figure 5C.2** Linear parametric model signals of Example 5.10.2.

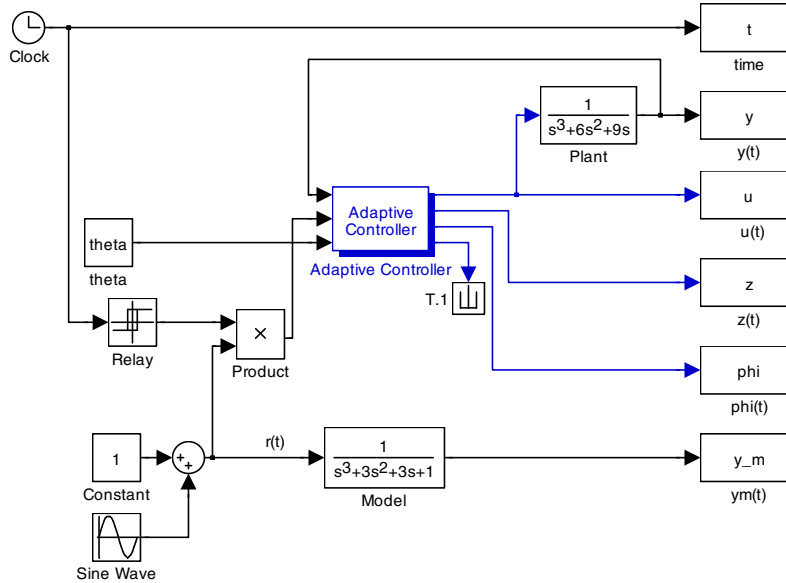


Figure 5C.3 Simulink scheme to generate the control signal in Example 5.10.2.

**Example 5.10.3** Consider, again, the plant and the reference model of Example 5.10.1. This time, let us assume that we don't know any information about  $\theta^*$ , except that  $0.1 \leq \theta_6^* = k_p/k_m \leq 10$ . Now we need to use a parameter identification algorithm to estimate  $\theta^*$ . For this task, we can modify the Simulink scheme of Figure 5C.3 as shown in Figure 5C.4. In this scheme, **Parameter Estimator** is incorporated into **Adaptive Controller**. The parameter identification algorithm is selected to be LS with forgetting factor  $\beta = 1$  and initial covariance  $P_0 = 10 I$ . Knowledge about  $\theta_6^*$  is used for parameter projection, and  $\theta$  is initialized as  $[0 \ 0 \ 0 \ 0 \ 0 \ 0.1]^T$ . The results are plotted in Figures 5C.5 and 5C.6. Although the parameter estimation is not successful, as seen in Figure 5C.6, Figure 5C.5 shows that control and tracking are successful.

Above, one may prefer to use the MATLAB function `umrcdr1` incorporated with one of the MATLAB functions for PI, e.g., `ucgrad` (see Chapter 3), instead of using Simulink as well. ■

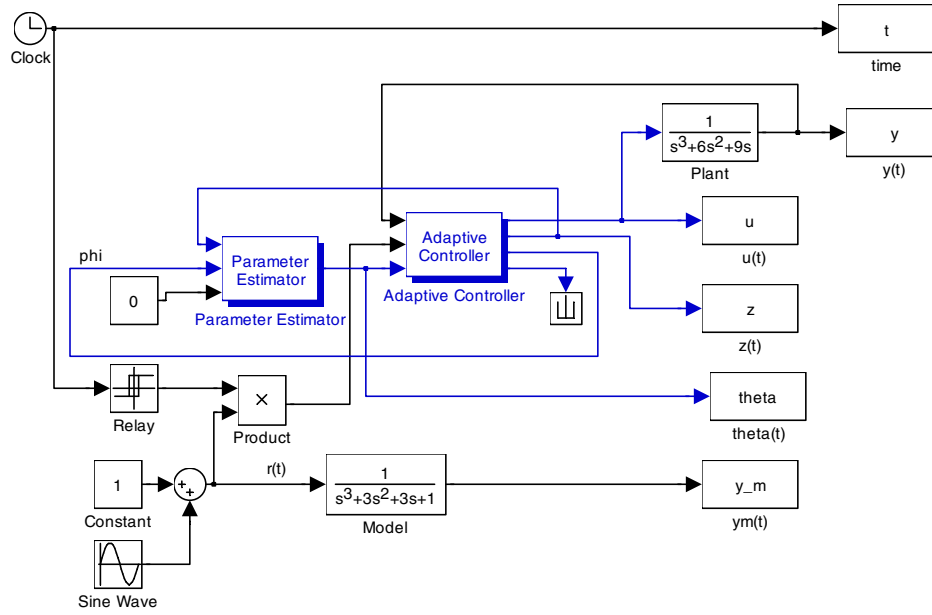


Figure 5C.4 Simulink scheme to generate the control signal in Example 5.10.3.

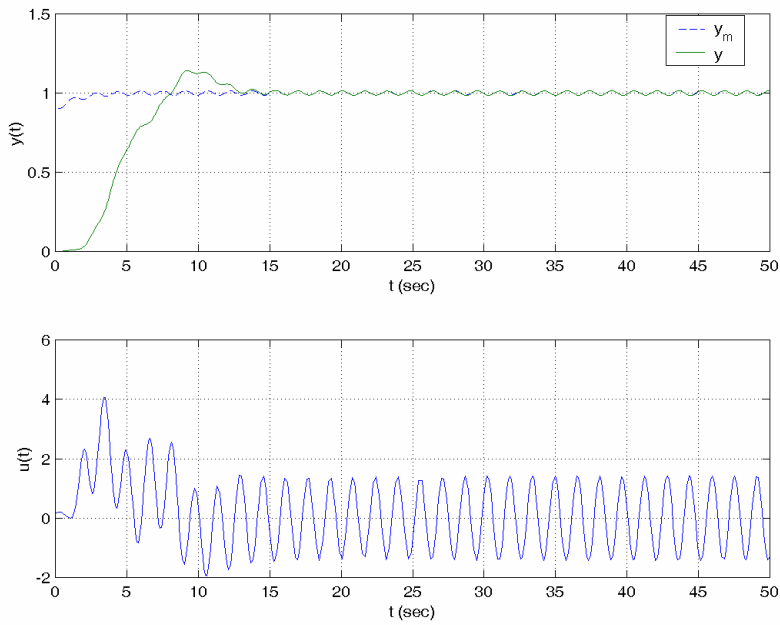
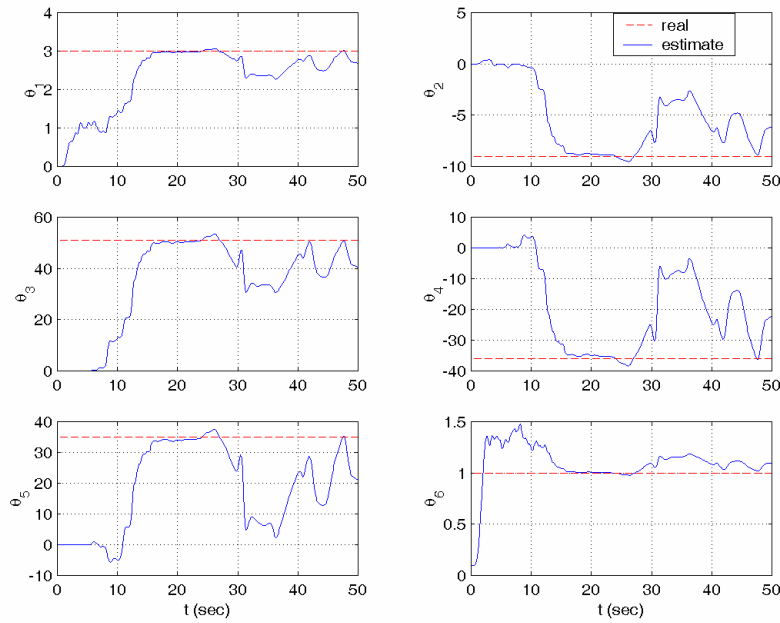


Figure 5C.5 Output and control signals of Example 5.10.3.





**Figure 5C.6** Linear model parameter estimates of Example 5.10.3.

**Example 5.10.4** Consider the scalar plant

$$\dot{x} = ax + bu,$$

where  $a, b$  are unknown parameters but  $b$  is known to be nonnegative. The objective is to choose an appropriate control law  $u$  such that all signals in the closed-loop system are bounded and  $x$  tracks the state  $x_m$  of the reference model given by

$$\dot{x}_m = -x_m + r$$

for a known reference signal  $r(t)$ .

The plant transfer function is  $G(s) = \frac{b}{s-a}$ , and the reference model transfer function is  $W_m(s) = \frac{1}{s+1}$ . So, we can construct the corresponding MRC as

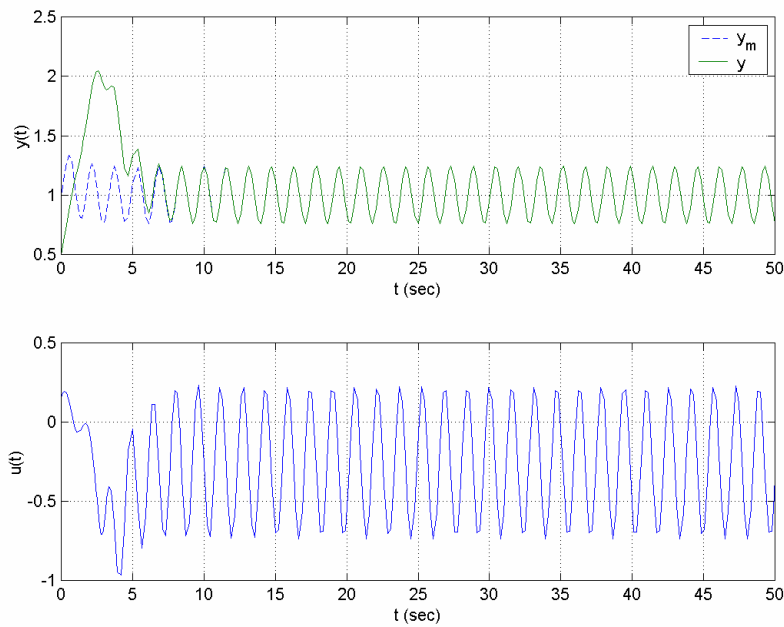
$$u = \theta_1^* x + \theta_2^* r,$$

since  $H(s) = \frac{\alpha(s)}{\lambda(s)} = 1$ . Here  $\theta_1^*$  and  $\theta_2^*$  can be estimated based on the SPM  $z = \theta^{*T} \phi$ , where

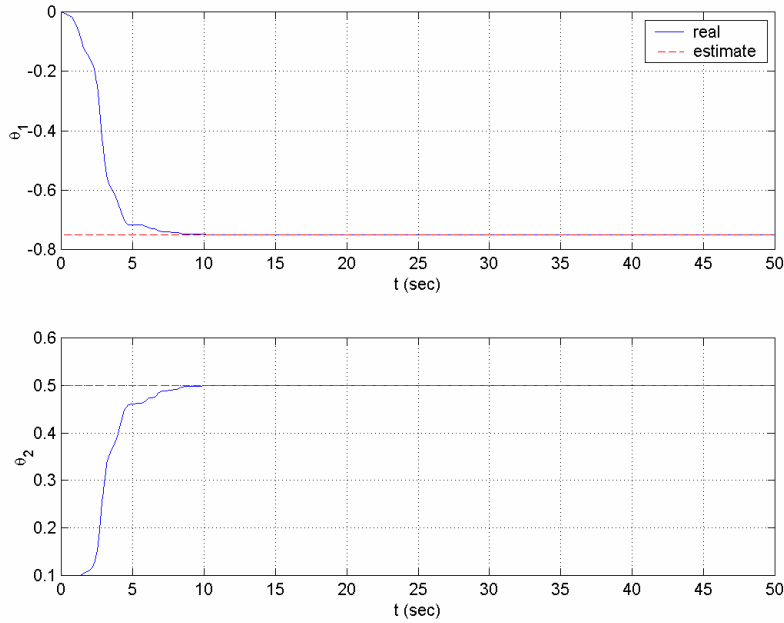
$$z = \frac{1}{s+1} u, \quad \phi = \left[ \frac{1}{s+1} x, x \right]^T.$$

Using  $z = \theta^{*T} \phi$  and  $x = \frac{b}{s-a} u$ , we can easily see that  $\theta_1^* = \frac{-a-1}{b}$  and  $\theta_2^* = \frac{1}{b}$ . Now let us simulate the plant, estimation of the parameters, and generation of the MRC signal for

$a = 0.5$ ,  $b = 2$ ,  $x_0 = 0.5$ , and  $r(t) = 1 + \sin(4t + \frac{\pi}{6})$ . The Simulink blocks **Parameter Estimator** and **Adaptive Controller** are used for simulation. The PI algorithm is selected to be LS with forgetting factor  $\beta = 1$  and  $P_0 = 10 I$ ,  $\theta$  is initialized as  $[0 \ 0.1]^T$ , and parameter projection for  $\theta_2^*$  is used assuming that  $0.1 \leq \theta_2^* \leq 10$ , as in Example 5.10.3. The results plotted in Figures 5C.7 and 5C.8 show that both the estimation and the tracking are successful. ■



**Figure 5C.7** Output and control signals of Example 5.10.4.



**Figure 5C.8** Linear model parameter estimates of Example 5.10.4.

**Example 5.10.5** This example is the bilinear counterpart of Example 5.10.2. The B-SPM for the system in Example 5.10.2 can be constructed as

$$z = \rho^* (\theta^{*T} \phi + z_1),$$

where

$$\theta^* = [\theta_u^{*T} \quad \theta_y^{*T} \quad \theta_r^{*T}]^T, \quad \rho^* = \frac{1}{\theta_r^*},$$

$$z = y - y_m, \quad y_m = W_m(s)r,$$

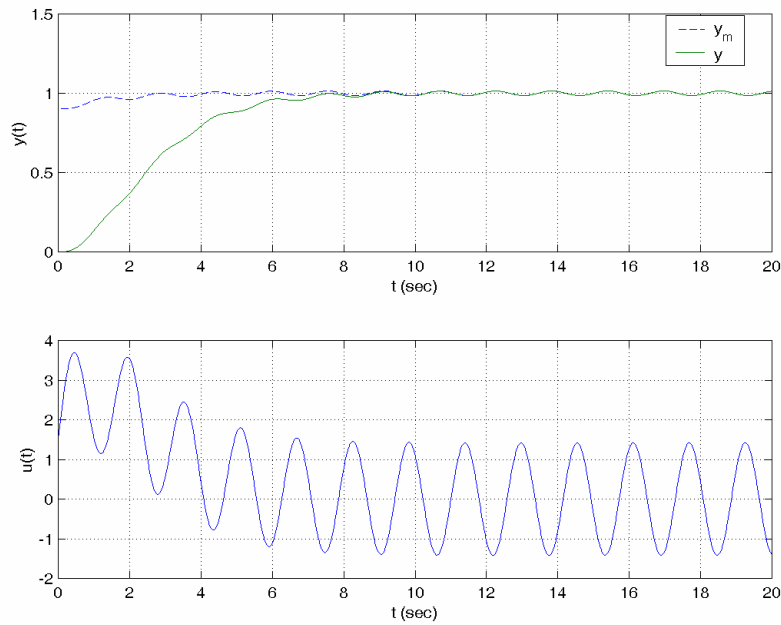
$$\phi = -[W_m(s)(H(s)u)^T, W_m(s)(H(s)y)^T, W_m(s)y, y_m]^T, \quad z_1 = W_m(s)u,$$

$$W_m(s) = \frac{1}{(s+1)^3}, \quad H(s) = \frac{\alpha(s)}{\Lambda(s)} = \frac{[s \quad 1]^T}{s^2 + s + 1}.$$

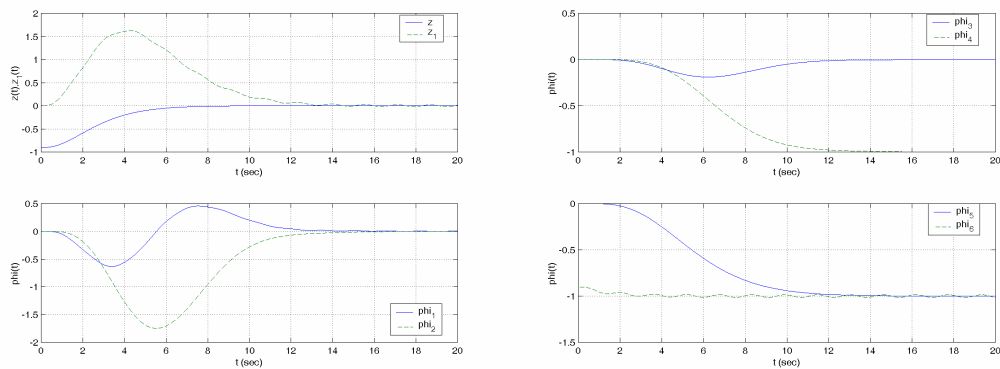
The control signal as well as the other signals in the B-SPM can be generated using the following lines instead of the corresponding lines in Example 5.10.2:

```
[nstate x10] = umrcdrb('init', [3 0], Lambda, Zm, Rm);
u(k) = umrcdrb('control', x1(:, k), [y(k) r(k)], [3
0], Lambda, Zm, Rm, theta);
[z(k), phi(:, k), z1(k)] = umrcdrb('output', x1(:, k), [u(k) y(k)
r(k)], ... [3 0], Lambda, Zm, Rm);
dx1 = umrcdrb('state', x1(:, k), [u(k) y(k) r(k)], Lambda, Zm, Rm);
```

The results are shown in Figures 5C.9 and 5C.10. One can use the Simulink block **Adaptive Controller** for simulation as well. ■

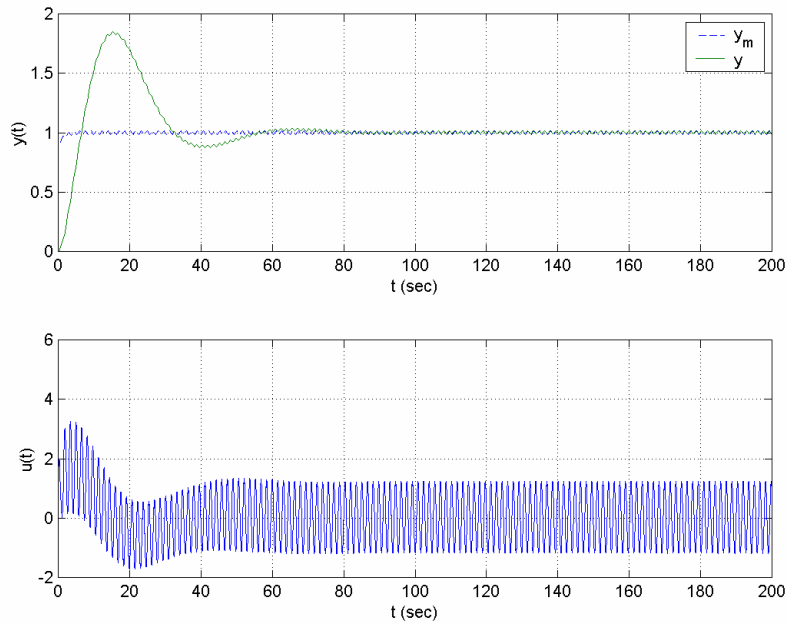


**Figure 5C.9** Output and control signals of Example 5.10.5.

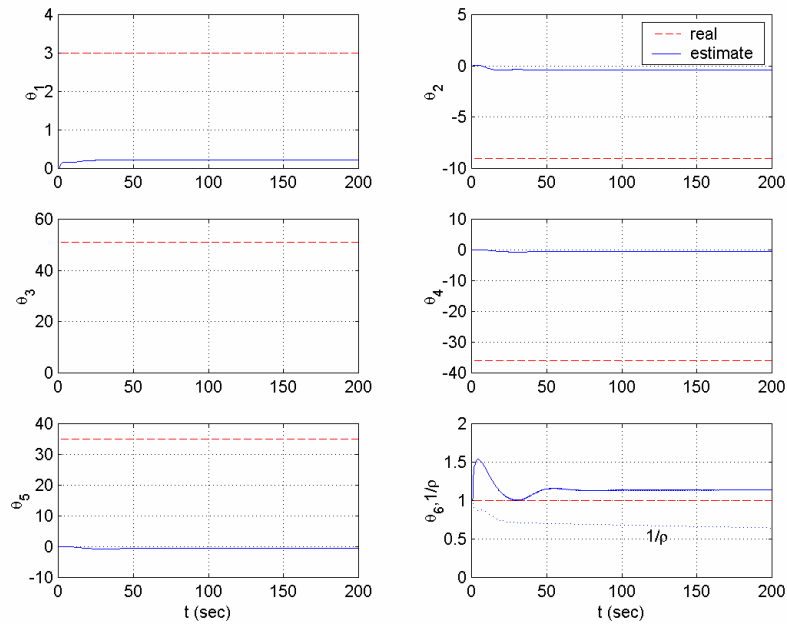


**Figure 5C.10** B-SPM signals in Example 5.10.5.

**Example 5.10.6** This example is the bilinear counterpart of Example 5.10.3. The B-SPM constructed in Example 5.10.5 is used for parameter estimation. Instead of the LS algorithm, a gradient algorithm with  $\Gamma = 100 \mathbf{I}$ ,  $\gamma = 100$ , and  $m_s = 1$  is used, and  $\theta_0(0) = 1/\rho(0)$  is selected to be 1. The results are shown in Figures 5C.11 and 5C.12. ■



**Figure 5C.11** Output and control signals in Example 5.10.6.



**Figure 5C.12** B-SPM parameter estimates of Example 5.10.6.

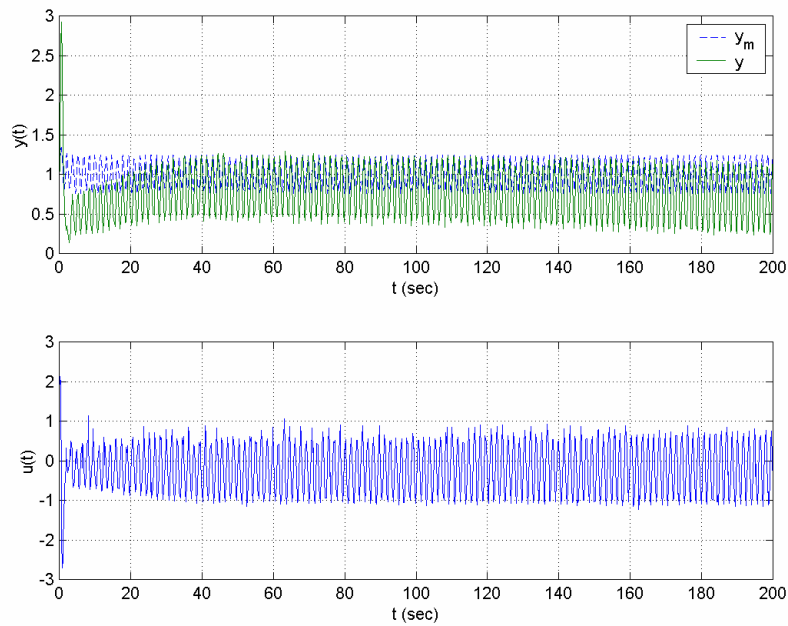
**Example 5.10.7** This example is the bilinear counterpart of Example 5.10.4. The B-SPM for the system considered in Example 5.10.4 can be constructed as  $z = \rho^* (\theta^{*T} \phi + z_1)$ , where

$$\theta^* = [\theta_1^* \quad \theta_2^*]^T, \quad \rho^* = \frac{1}{\theta_2^*},$$

$$z = x - x_m, \quad x_m = \frac{1}{s+1} r,$$

$$\phi = -\left[ \frac{1}{s+1} x, x_m \right]^T, \quad z_1 = \frac{1}{s+1} u.$$

Instead of the LS algorithm of Example 5.10.4, a gradient algorithm with  $\Gamma = 100 \mathbf{I}$ ,  $\gamma = 100$ , and  $m_s = 1$  is used, and  $\theta_2(0) = 1/\rho(0)$  is chosen to be 1. The results are shown in Figures 5C.13 and 5C.14. ■



**Figure 5C.13** Output and control signals of Example 5.10.7.

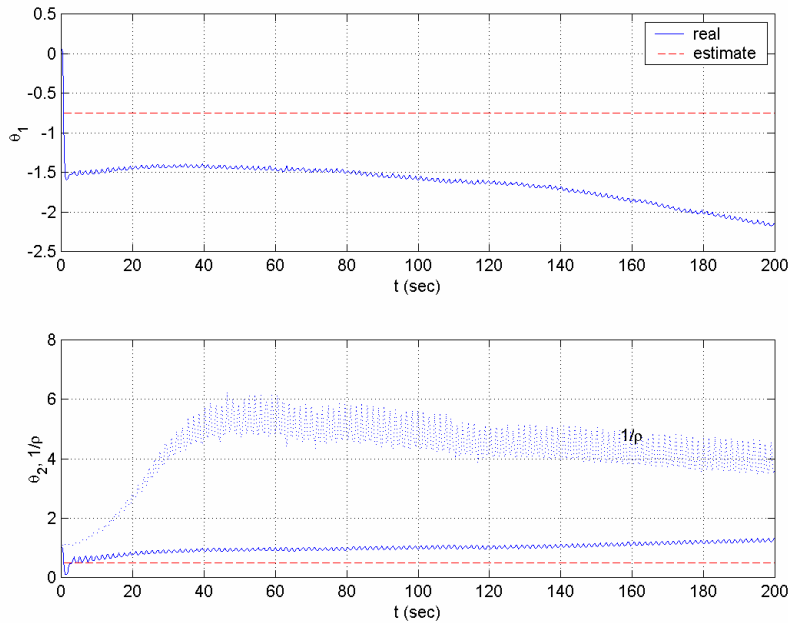
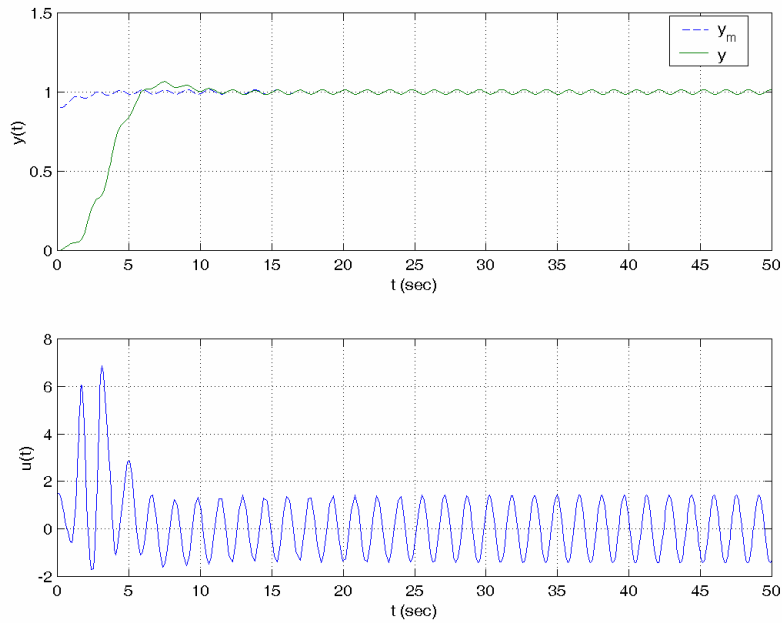


Figure 5C.14 Linear model parameter estimates of Example 5.10.7.

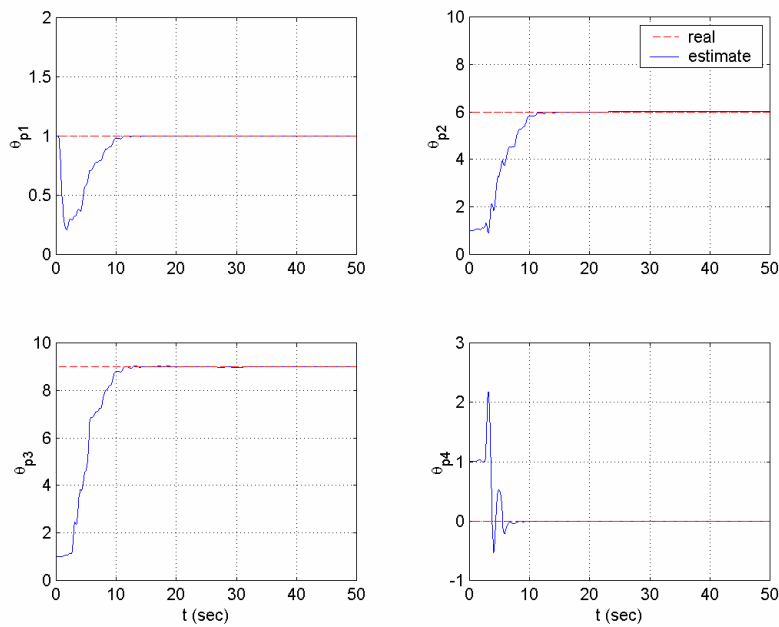
### 5.10.3 Indirect MRAC

The indirect MRAC schemes of section 5.7 can be implemented using the MATLAB functions `mrcpoly`, `umrcidr` or the Simulink block **Adaptive Controller** of the Adaptive Control Toolbox, incorporated with the PI functions (or the Simulink block **Parameter Estimator**) described in Chapter 3. As mentioned before, each indirect MRAC scheme is composed of a PI algorithm to generate the estimates of the plant parameters, and a control law to generate the control signal using these plant parameter estimates. The task of the function `mrcpoly` is to calculate the control parameters in this control law based on (5.133). `umrcidr` generates the control signal as well as the other signals involved in the parametric model used in PI. The uses of the indirect MRAC tools of the Adaptive Control Toolbox are demonstrated in the following examples.

**Example 5.10.8** Consider the plant and the reference model of Example 5.10.1. Let us assume that we don't know any information about the plant, except that  $0.1 \leq k_p/k_m \leq 10$ . We want to perform the control task of Example 5.10.3 using the indirect MRAC. We can use the Simulink scheme shown in Figure 5C.4, changing only the entries of the **Adaptive Controller** ( $\Lambda_p(s)$  is selected to be  $\Lambda_p(s) = (s+2)\Lambda(s)$ ) and the **Parameter Estimator** blocks. The PI algorithm is again selected to be LS with forgetting factor  $\beta = 1$  and  $P_0 = 10 \mathbf{I}$ .  $\theta$  is initialized as  $\theta_0 = [1, 1, 1, 1]^T$ , and parameter projection is not used. The results are plotted in Figures 5C.15 and 5C.16.



**Figure 5C.15** Output and control signals of Example 5.10.8.



**Figure 5C.16** SPM parameter estimate  $\theta = [\hat{b}_0 \quad \hat{a}_2 \quad \hat{a}_1 \quad \hat{a}_0]^T$  of Example 5.10.8.

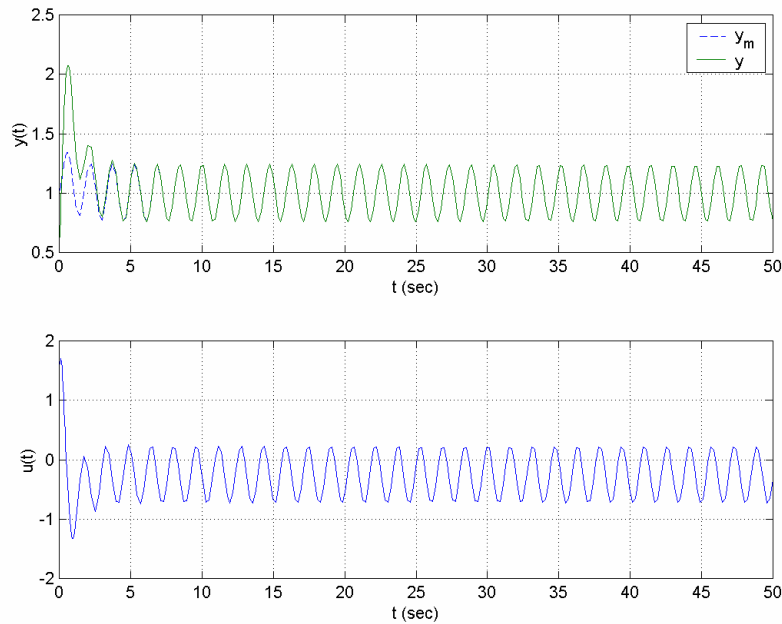
As above, one may prefer to use the MATLAB function `umrcidr` incorporated with one of the MATLAB functions for parameter identification, e.g., `ucgrad` (see Chapter 3), instead of using Simulink. ■



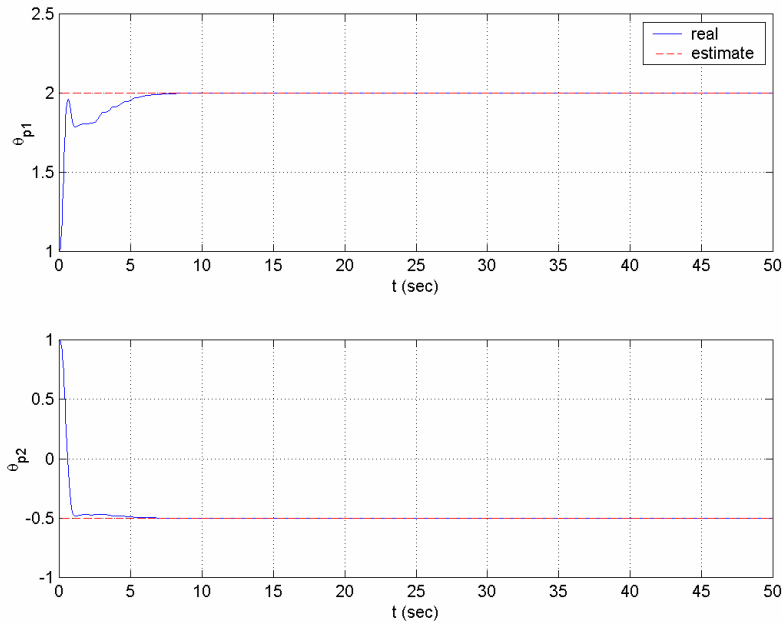
**Example 5.10.9** Consider the scalar plant of Example 5.10.4 with the same reference signal. We want to perform the same control task using indirect MRAC. Choosing  $\Lambda_p(s) = s + 2$ , parameter estimation is based on the linear model  $z = \theta^{*T} \phi$ , where

$$z = \frac{s}{s+2} y, \quad \theta_p^* = [b \quad -a]^T, \quad \phi = \left[ \frac{1}{s+2} u, \quad -\frac{1}{s+2} y \right]^T.$$

The Simulink blocks **Parameter Estimator** and **Adaptive Controller** are used for simulation. The PI algorithm is selected to be LS with forgetting factor  $\beta = 1$  and  $P_0 = 10 I$ ,  $\theta$  is initialized as  $\theta_0 = [1, 1]^T$ , and parameter projection is not used. The results plotted in Figures 5C.17 and 5C.18 show that both the estimation and the tracking are successful. ■



**Figure 5C.17** Output and control signals of Example 5.10.9.



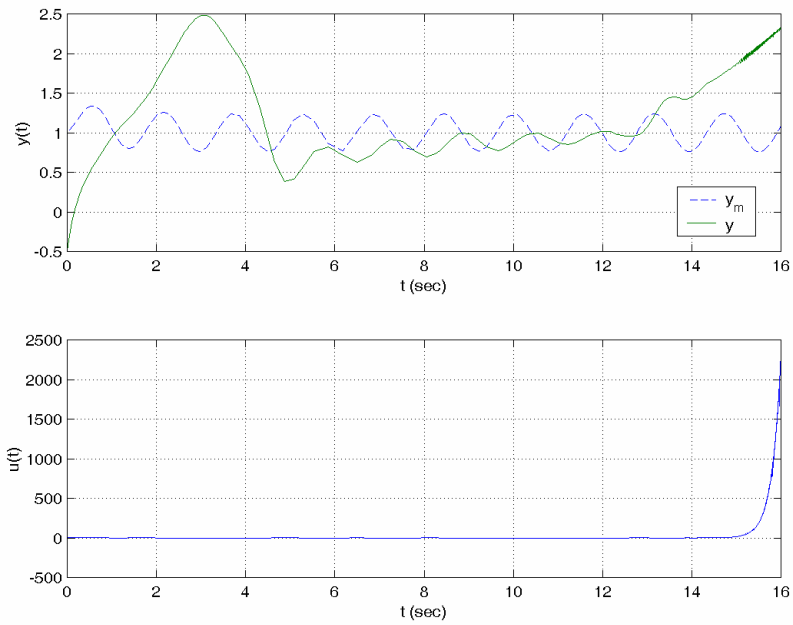
**Figure 5C.18** SPM parameter estimate  $\theta = [\hat{b}_1 \quad \hat{a}_0]^T$  in Example 5.10.9.

### 5.10.4 Robust MRAC

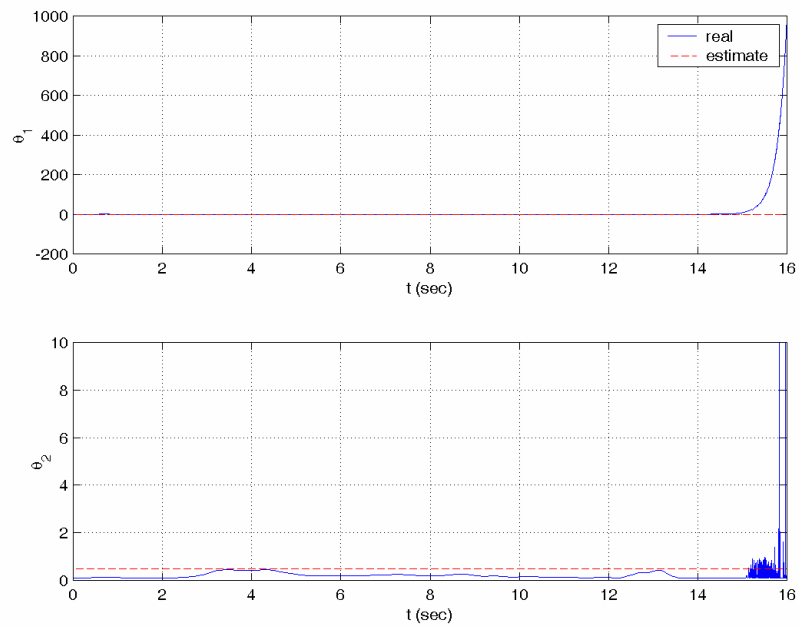
**Example 5.10.10** Consider the scalar plant of Example 5.10.4. In this example we will consider the existence of a multiplicative plant disturbance  $\Delta_m(s) = \frac{-2\mu s}{1+\mu s}$ ; i.e., we will assume that

$$x = \frac{b}{s-a} (1 + \Delta_m(s)) u = \frac{b}{s-a} \frac{1-\mu s}{1+\mu s} u.$$

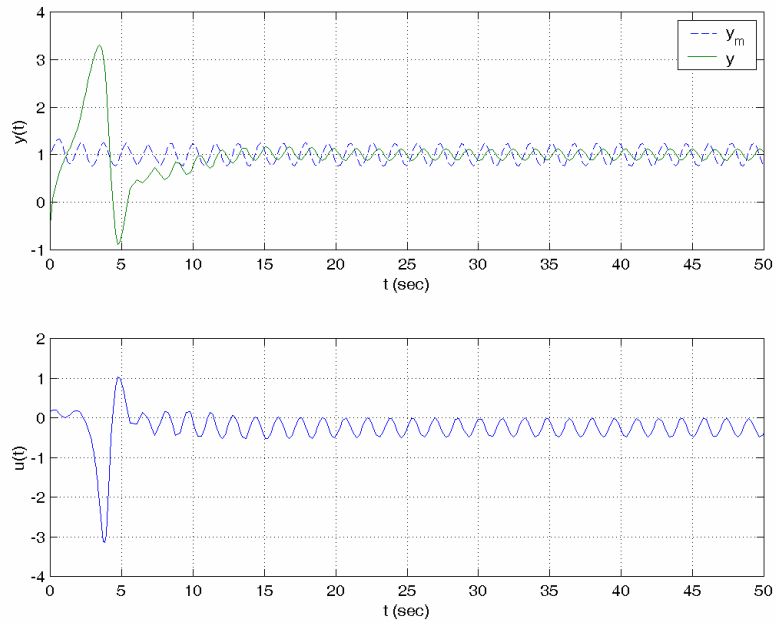
Let  $a = 0.5$ ,  $b = 2$ ,  $x_0 = 0.5$ , and  $r(t) = 1 + \sin(4t + \frac{\pi}{6})$  as in Example 5.10.4, and let  $\mu = 0.2$ . If we use the PI algorithm and the MRC law of Example 5.10.4 as they are, we obtain the results shown in Figures 5C.19 and 5C.20. As can be seen in these figures,  $\theta_1$  and  $x$  diverge, and the adaptive control design fails. Next, we apply the robust PI algorithm with dead zone modification ( $g_0 = 0.05, m_s = 1$ ). The results plotted in Figures 5C.21 and 5C.22 show that the parameter estimates do not diverge and the tracking error remains in an allowable range. ■



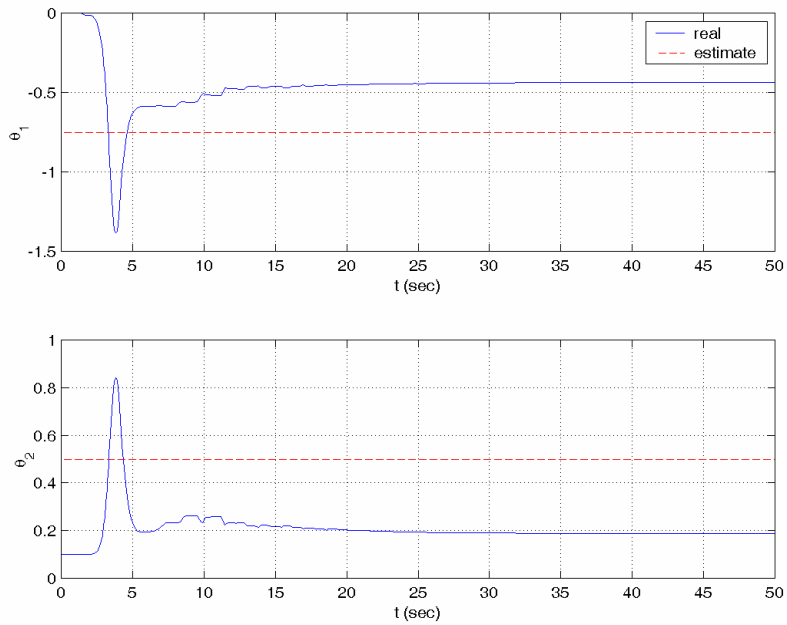
**Figure 5C.19** Output and control signals of Example 5.10.10 without robustness modification.



**Figure 5C.20** SPM parameters of Example 5.10.10 without robustness modification.



**Figure 5C.21** Output and control signals of Example 5.10.10 with robustness modification.



**Figure 5C.22** SPM parameter estimates of Example 5.10.10 with robustness modification.

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