

Chapter 6

Complementary Material

6.3 APPC Schemes: Polynomial Approach

Proof of Theorem 6.3.2.

Step 1. We establish the expressions (6.48) We rewrite the control law (6.39) and the normalized estimation error as

$$\begin{aligned}\hat{L}Q_m \frac{1}{\Lambda} u_p &= -\hat{P} \frac{1}{\Lambda} (y_p - y_m), \\ \varepsilon m_s^2 &= z - \theta_p^\top \phi = \hat{R}_p \frac{1}{\Lambda_p} y_p - \hat{Z}_p \frac{1}{\Lambda_p} u_p.\end{aligned}\tag{6C.1}$$

$\Lambda(s), \Lambda_p(s)$ are monic Hurwitz polynomials of degree $n + q - 1$, n , respectively,

$$\hat{R}_p = s^n + \theta_a^\top \alpha_{n-1}(s), \quad \hat{Z}_p = \theta_b^\top \alpha_{n-1}(s).\tag{6C.2}$$

The proof is simplified if without loss of generality we design

$$\Lambda(s) = \Lambda_p(s) \Lambda_q(s),\tag{6C.3}$$

where $\Lambda_q(s)$ is an arbitrary monic Hurwitz polynomial of degree $q - 1$ if $q \geq 2$ and $\Lambda_q(s) = 1$ for $q < 2$. We should point out that the same analysis can also be carried out with Λ, Λ_p being Hurwitz but otherwise arbitrary, at the expense of some additional algebra. We leave this more general case as an exercise for the reader.

Let us define

$$u_f \triangleq \frac{1}{\Lambda} u_p, \quad y_f \triangleq \frac{1}{\Lambda} y_p\tag{6C.4}$$

and write (6C.1) as

$$\begin{aligned}\hat{P} y_f + \hat{L} Q_m u_f &= y_{m1}, \\ \hat{R}_p \Lambda_q y_f - \hat{Z}_p \Lambda_q u_f &= \varepsilon m_s^2,\end{aligned}\tag{6C.5}$$

where

$$y_{m1} \triangleq \hat{P} \frac{1}{\Lambda} y_m \in L_\infty.\tag{6C.6}$$

Use the expressions

$$\hat{R}_p(s)\Lambda_q(s) = s^{n+q-1} + \bar{\theta}_1^\top \alpha_{n+q-2}(s), \quad \hat{Z}_p(s)\Lambda_q(s) = \bar{\theta}_2^\top \alpha_{n+q-2}(s), \quad (6C.7)$$

$$\hat{P}(s) = p_0 s^{n+q-1} + \bar{p}^\top \alpha_{n+q-2}(s), \quad \hat{L}(s)Q_m(s) = s^{n+q-1} + \bar{l}^\top \alpha_{n+q-2}(s) \quad (6C.8)$$

in (6C.5) to obtain

$$\begin{aligned} y_f^{(n+q-1)} &= -\bar{\theta}_1^\top \alpha_{n+q-2}(s)y_f + \bar{\theta}_2^\top \alpha_{n+q-2}(s)u_f + \varepsilon m_s^2, \\ u_f^{(n+q-1)} &= (p_0 \bar{\theta}_1 - \bar{p})^\top \alpha_{n+q-2}(s)y_f - (p_0 \bar{\theta}_2 + \bar{l})^\top \alpha_{n+q-2}(s)u_f - p_0 \varepsilon m_s^2 + y_{m1}. \end{aligned} \quad (6C.9)$$

Define the state

$$x \triangleq \left[y_f^{(n+q-2)}, \dots, \dot{y}_f, y_f, u_f^{(n+q-2)}, \dots, \dot{u}_f, u_f \right]^\top.$$

Then (6C.9) can be expressed in the form

$$\dot{x} = A(t)x + b_1(t)\varepsilon m_s^2 + b_2 y_{m1}, \quad (6C.10)$$

where

$$A(t) = \left[\begin{array}{c|c} -\bar{\theta}_1^\top & \bar{\theta}_2^\top \\ \hline I_{n+q-2} \begin{array}{c} \vdots \\ 0 \end{array} & O_{(n+q-2) \times (n+q-1)} \\ \hline p_0 \bar{\theta}_1^\top - \bar{p}^\top & p_0 \bar{\theta}_2^\top - \bar{l}^\top \\ \hline O_{(n+q-2) \times (n+q-1)} & I_{n+q-2} \begin{array}{c} \vdots \\ 0 \end{array} \end{array} \right], \quad (6C.11)$$

where $O_{(n+q-2) \times (n+q-1)}$ is an $(n+q-2) \times (n+q-1)$ matrix with all elements equal to zero. Since

$$\begin{aligned} u_p &= \Lambda u_f = u_f^{(n+q-1)} + \lambda^\top \alpha_{n+q-2}(s)u_f, \\ y_p &= \Lambda y_f = y_f^{(n+q-1)} + \lambda^\top \alpha_{n+q-2}(s)y_f, \end{aligned} \quad (6C.12)$$

where λ is the coefficient vector of $\Lambda(s) - s^{n+q-1}$, it follows that

$$\begin{aligned} u_p &= \underbrace{[0, \dots, 0]_{n+q-1}}_{n+q-1} \underbrace{[1, 0, \dots, 0]_{n+q-1}}_{n+q-1} \dot{x} + \underbrace{[0, \dots, 0]_{n+q-1}}_{n+q-1} \lambda^\top x, \\ y_p &= \underbrace{[1, 0, \dots, 0]_{n+q-1}}_{n+q-1} \underbrace{[0, \dots, 0]_{n+q-1}}_{n+q-1} \dot{x} + \lambda^\top \underbrace{[0, \dots, 0]_{n+q-1}}_{n+q-1} x. \end{aligned} \quad (6C.13)$$

Step 2. Establish the e.s. of the homogeneous part of (6C.11) We can show that for each frozen time t

$$\det(sI - A(t)) = \hat{R}_p \Lambda_q \cdot \hat{L} Q_m + \hat{P} \cdot \hat{Z}_p \Lambda_q = A^* \Lambda_q, \quad (6C.14)$$

where the last equality is obtained by using the Diophantine equation (6.38). Since Λ_q, A^* are Hurwitz, $A(t)$ has eigenvalues with negative real parts at each time t ; i.e., $A(t)$ is stable at each frozen time t .

We now need to show that $\dot{A}(t) \in L_2, A(t) \in L_\infty$ from the properties $\theta_p \in L_\infty$ and $\dot{\theta}_p \in L_2$, which are guaranteed by the adaptive law (6.37). In a similar way as in the known parameter case, the matching equation (6.38) may be expressed as

$$\hat{S}_l \hat{\beta}_l = a_l^*, \quad (6C.15)$$

where \hat{S}_l is the Sylvester matrix of $\hat{R}_p Q_m, \hat{Z}_p$; a_l^* is as defined in (6.32); and

$$\hat{\beta}_l = [\hat{l}_q, \hat{p}]^T, \quad \hat{l}_q = [\underbrace{0 \dots 0}_q, 1, \hat{l}]^T.$$

Using the assumption that the polynomials $\hat{R}_p Q_m$ and \hat{Z}_p are strongly coprime at each time t , we conclude that their Sylvester matrix \hat{S}_l is uniformly nonsingular; i.e., $|\det(\hat{S}_l)| > \nu_0$ for some $\nu_0 > 0$. Therefore, $\theta_p \in L_\infty$ implies that $S_l, S_l^{-1} \in L_\infty$. From (6C.15) we have

$$\hat{\beta}_l = \hat{S}_l^{-1} a_l^*, \quad (6C.16)$$

which implies that $\hat{\beta}_l$ is u.b. On the other hand, because $\theta_p \in L_\infty$ and $\dot{\theta}_p \in L_2$, it follows from the definition of the Sylvester matrix that $\hat{S}_l \in L_2$. Noting that

$$\hat{\dot{\beta}}_l = -\hat{S}_l^{-1} \dot{\hat{S}}_l \hat{S}_l^{-1} a_l^*,$$

we have $\hat{\dot{\beta}}_l \in L_2$, which is implied by $\hat{S}_l, \hat{S}_l^{-1} \in L_\infty$ and $\dot{\hat{S}}_l(t) \in L_2$. Because the vectors $\bar{\theta}_1, \bar{\theta}_2, \bar{p}, \bar{l}$ are linear combinations of $\theta_p, \hat{\beta}_l$ and all elements in $A(t)$ are u.b., we have

$$\|\dot{A}(t)\| \leq c(|\hat{\dot{\beta}}_l(t)| + |\dot{\theta}_p(t)|),$$

which implies that $\dot{A}(t) \in L_2$. Using Theorem A.8.6, it follows that the homogeneous part of (6C.10) is e.s.

Step 3. Use the properties of the $L_{2\sigma}$ norm and B–G lemma to establish boundedness For clarity of presentation let us denote the $L_{2\delta}$ norm as $\|\cdot\|$. From Lemma A.5.10 and (6C.10) we have

$$\|x\| \leq c\epsilon m_s^2 + c, \quad |x(t)| \leq c\epsilon m_s^2 + c \quad (6C.17)$$

for some $\delta > 0$. We define the fictitious signal

$$m_f^2 \triangleq 1 + y_p^2 + u_p^2. \quad (6C.18)$$

Using (6C.10), (6C.12), (6C.17) and Lemma A.5.10 we can establish that $|\phi|, |m_s| \leq m_f$ and

$$m_f^2 \leq c + c \|x^2\| + c \|\varepsilon m_s^2\|^2 \leq c \|\varepsilon m_s m_f\|^2 + c, \quad (6C.19)$$

i.e.,

$$m_f^2 \leq c + c \int_0^t e^{-\delta(t-\tau)} \varepsilon^2(\tau) m_s^2(\tau) m_f^2(\tau) d\tau.$$

Using the property $\varepsilon m_s \in L_2$, guaranteed by the adaptive law, using the boundedness of m_f , and applying the B-G lemma, we can establish that $m_f \in L_\infty$. The boundedness of the rest of the signals follows from $m_f \in L_\infty$ and the properties of the $L_{2\delta}$ norm that is used to show that m_f bounds most of the signals from above.

Step 4. Establish that the tracking error converges to zero Let us start with the second equation in (6C.1), i.e.,

$$\varepsilon m_s^2 = \hat{R}_p \frac{1}{\Lambda_p} y_p - \hat{Z}_p \frac{1}{\Lambda_p} u_p.$$

Filtering each side of the above equation with $\bar{L}Q_m \frac{1}{\Lambda_q}$, where

$$\bar{L}(s, t) \triangleq s^{n-1} + \alpha_{n-2}^\top(s)l$$

and $l_c = [1, t^\top]^\top$ is the coefficient vector of $\hat{L}(s, t)$, we obtain

$$\bar{L}Q_m \frac{1}{\Lambda_q} (\varepsilon m_s^2) = \bar{L}Q_m \frac{1}{\Lambda_q} \left(\hat{R}_p \frac{1}{\Lambda_p} y_p - \hat{Z}_p \frac{1}{\Lambda_p} u_p \right). \quad (6C.20)$$

Using $\Lambda = \Lambda_q \Lambda_p$, and applying Lemma A.11.1 (Swapping Lemma 1), we obtain the following:

$$\frac{Q_m}{\Lambda_q} \hat{R}_p \frac{1}{\Lambda_p} y_p = \hat{R}_p \frac{Q_m}{\Lambda} y_p + r_1, \quad \frac{Q_m}{\Lambda_q} \left(\hat{Z}_p \frac{1}{\Lambda_p} u_p \right) = \hat{Z}_p \frac{Q_m}{\Lambda} u_p + r_2, \quad (6C.21)$$

where

$$r_1 \triangleq W_{c1}(s) \left(\left[W_{b1}(s) \frac{\alpha_{n-1}^\top(s)}{\Lambda_p} y_p \right] \dot{\theta}_a \right), \quad r_2 \triangleq W_{c1}(s) \left(\left[W_{b1}(s) \frac{\alpha_{n-1}^\top(s)}{\Lambda_p} u_p \right] \dot{\theta}_b \right), \quad (6C.22)$$

and W_{c1}, W_{b1} are as defined in Swapping Lemma 1 with $W = \frac{Q_m}{\Lambda_q}$.

Because $u_p, y_p \in L_\infty$ and $\dot{\theta}_a, \dot{\theta}_b \in L_2$, it follows that $r_1, r_2 \in L_2$. Using (6C.21) in (6C.20), we obtain

$$\bar{L}Q_m \frac{1}{\Lambda_q} (\varepsilon m_s^2) = \bar{L} \left(\hat{R}_p \frac{Q_m}{\Lambda} y_p - \hat{Z}_p \frac{Q_m}{\Lambda} u_p + r_1 - r_2 \right) \quad (6C.23)$$

which, due to $Q_m y_p = Q_m(e_1 + y_m) = Q_m e_1$, becomes

$$\bar{L}Q_m \frac{1}{\Lambda_q}(\varepsilon m_s^2) = \bar{L} \left[\hat{R}_p \frac{Q_m}{\Lambda} e_1 + r_1 - r_2 \right] - \bar{L} \left[\hat{Z}_p \frac{Q_m}{\Lambda} u_p \right]. \quad (6C.24)$$

Applying Lemma A.11.4 (Swapping Lemma 4) (i) to the second term in the right-hand side of (6C.24), we obtain

$$\bar{L} \left[\hat{Z}_p \frac{Q_m}{\Lambda} u_p \right] = \bar{Z}_p \left[\hat{L}(s,t) \frac{Q_m}{\Lambda} u_p \right] + r_3, \quad (6C.25)$$

where

$$\bar{Z}_p = \alpha_{n-1}^T(s)\theta_b, \quad r_3 = \alpha_{n-2}^T(s)F(l,\theta_b)\alpha_{n-2}(s) \frac{Q_m}{\Lambda(s)} u_p,$$

and $\|F(l,\theta_b)\| \leq c(|\dot{l}| + |\dot{\theta}_b|)$. Using the identity $\hat{L}(s,t)Q_m(s) = \hat{L}(s,t)Q_m(s)$ and the control law given by the first equation in (6C.1), we obtain

$$\hat{L}(s,t) \frac{Q_m}{\Lambda} u_p = \hat{L}(s,t) \frac{Q_m}{\Lambda} u_p = -\hat{P} \frac{1}{\Lambda} e_1. \quad (6C.26)$$

From (6C.26), (6C.25), (6C.24) we obtain

$$\bar{L}Q_m \frac{1}{\Lambda_q}(\varepsilon m_s^2) = \bar{L} \left[\hat{R}_p \frac{Q_m}{\Lambda} e_1 \right] + \bar{Z}_p \left[\hat{P} \frac{1}{\Lambda} e_1 \right] + \bar{L}(r_1 - r_2) - r_3. \quad (6C.27)$$

Using Lemma A.11.4(ii) with $f = 1/\Lambda, \Lambda_0 = 1$, we can obtain the expressions

$$\begin{aligned} \bar{L} \left[\hat{R}_p \frac{Q_m}{\Lambda} e_1 \right] &= \overline{\hat{L}(s,t) \cdot \hat{R}_p(s,t) Q_m} \frac{1}{\Lambda} e_1 + r_4, \\ \bar{Z}_p \left[\hat{P} \frac{1}{\Lambda} e_1 \right] &= \overline{\hat{Z}_p(s,t) \cdot \hat{P}(s,t)} \frac{1}{\Lambda} e_1 + r_5, \\ r_4 &\triangleq \alpha_{n-1}^T(s)G(s, e_1, l, \theta_a), \quad r_5 \triangleq \alpha_{n-1}^T(s)G'(s, e_1, \theta_b, p), \end{aligned} \quad (6C.28)$$

where G, G' are as defined in Lemma A.11.4 and $\overline{X(s,t)} \triangleq \alpha_n^T(s)\theta(t)$ denotes the swapped polynomial $X(s,t) \triangleq \theta^T(t)\alpha_n(s)$. Due to $e_1 \in L_\infty$ and $\dot{l}, \dot{p}, \dot{\theta}_a, \dot{\theta}_b \in L_2$ it follows from the expression of G, G' that $G, G' \in L_2$. Using (6C.28) in (6C.27), we obtain

$$\bar{L}Q_m \frac{1}{\Lambda_q}(\varepsilon m_s^2) = \left(\overline{\hat{L}(s,t) \hat{R}_p(s,t) Q_m} + \overline{\hat{Z}_p(s,t) \hat{P}(s,t)} \right) \frac{1}{\Lambda} e_1 + \bar{L}(r_1 - r_2) - r_3 + r_4 + r_5. \quad (6C.29)$$

Using the matching equation (6.38) and the fact that $\overline{A^*(s)} = A^*(s)$ due to the constant coefficients of $A^*(s)$, we can write

$$\overline{\hat{L}(s,t) \cdot \hat{R}_p(s,t) Q_m} + \overline{\hat{Z}_p(s,t) \cdot \hat{P}(s,t)} = \overline{A^*(s)} = A^*(s). \quad (6C.30)$$

Substituting (6C.30) in (6C.29), we obtain

$$A^* \frac{1}{\Lambda} e_1 = v, \quad v \triangleq \bar{L} Q_m \frac{1}{\Lambda_q} (\varepsilon m_s^2) - \bar{L}(r_1 - r_2) + r_3 - r_4 - r_5 \quad (6C.31)$$

or

$$e_1 = \frac{\Lambda}{A^*} v. \quad (6C.32)$$

Using the fact that $\bar{L}(s) = \alpha_{n-1}^T(s) l_c$, $l_c = [1, l^T]^T$, we have

$$v = \alpha_{n-1}^T(s) l_c \left[\frac{Q_m}{\Lambda_q} \varepsilon m_s^2 - r_1 + r_2 \right] + r_3 - r_4 - r_5.$$

Therefore, (6C.32) may be written as

$$\begin{aligned} e_1 &= \frac{\Lambda}{A^*} \alpha_{n-1}^T(s) l_c \frac{Q_m}{\Lambda_q} \varepsilon m_s^2 + w_0, \\ w_0 &= \frac{\Lambda}{A^*} [\alpha_{n-1}^T(s) l_c (r_2 - r_1) + \alpha_{n-2}^T(s) w_1 - \alpha_{n-1}^T(s) w_2], \\ w_1 &= F(l, \theta_b) \alpha_{n-2}(s) \frac{Q_m}{\Lambda} u_p, \\ w_2 &= G + G'. \end{aligned} \quad (6C.33)$$

Since $l_c, u_p \in L_\infty$; $F, G, G', r_1, r_2 \in L_2$; and $\frac{\Lambda}{A^*} \alpha_{n-1}^T(s), \alpha_{n-2}(s) \frac{Q_m}{A^*}$ are strictly proper and stable, it follows from Corollary A.5.5 that $w_0, w_1, w_2 \in L_2$ and $w_0 \rightarrow 0$ as $t \rightarrow \infty$. Let us now apply Lemma A.11.1 (Swapping Lemma 1) to the first equation in (6C.33) to obtain

$$e_1 = l_c^T \frac{\alpha_{n-1} \Lambda}{A^* \Lambda_q} \varepsilon m_s^2 + W_c(s) [W_b(s) \varepsilon m_s^2] l_c + w_0, \quad (6C.34)$$

where $W_c(s), W_b(s)$ consist of strictly proper stable transfer functions. Since all the transfer functions in (6C.34) are proper with inputs which are bounded and in L_2 , it follows that $e_1 \in L_\infty \cap L_2$. We can write the plant equation $R_p y_p = Z_p u_p$ as

$$\Lambda_p y_p + (R_p - \Lambda_p) y_p = Z_p u_p$$

or

$$y_p = -\frac{(R_p - \Lambda_p)}{\Lambda_p} y_p + Z_p u_p, \quad \dot{y}_p = -\frac{s(R_p - \Lambda_p)}{\Lambda_p} y_p + s Z_p u_p.$$

Since Λ_p is monic, stable with the same degree as R_p , all the transfer functions in the above equations are proper, and the boundedness of y_p, u_p implies that \dot{y}_p is bounded. In turn, this implies that \dot{e}_1 is bounded, which together with $e_1 \in L_\infty \cap L_2$ implies that $e_1 \rightarrow 0$ as $t \rightarrow \infty$. \square

6.4 APPC Schemes : State-Space Approach

Proof of Theorem 6.4.3.

Step 1. Develop the closed-loop state error equations We express the tracking error equation

$$e_1 = \frac{Z_p Q_1}{R_p Q_m} \bar{u}_p$$

in the state-space form

$$\dot{e} = \begin{bmatrix} -\theta_1^* & \left| \frac{I_{n+q-1}}{0} \right. \\ \hline \end{bmatrix} e + \theta_2^* \bar{u}_p, \quad e_1 = C^T e, \quad (6C.35)$$

where $e \in R^{n+q}$, $C^T = [1, 0, \dots, 0] \in R^{n+q}$, and θ_1^*, θ_2^* are the coefficient vectors of $R_p Q_m - s^{n+q}, Z_p Q_1$, respectively.

Let $e_o \triangleq e - \hat{e}$ be the state-observation error. Using (6.67) and (6C.35), we obtain

$$\begin{aligned} \dot{\hat{e}} &= A_c(t) \hat{e} + \hat{K}_o C^T e_o, \\ \dot{e}_o &= A_o e_o + \tilde{\theta}_1 e_1 - \tilde{\theta}_2 \bar{u}_p, \end{aligned} \quad (6C.36)$$

where

$$A_o \triangleq \begin{bmatrix} -a^* & \left| \frac{I_{n+q-1}}{0} \right. \\ \hline \end{bmatrix}$$

is a stable matrix; $A_c(t) \triangleq \hat{A} - \hat{B} \hat{K}_c$; and $\tilde{\theta}_1 \triangleq \theta_1 - \theta_1^*, \tilde{\theta}_2 \triangleq \theta_2 - \theta_2^*$. Since the polynomials $Z_p Q_1, R_p Q_m$ have no common unstable zeros it follows using Theorem A.2.5 that there exist polynomials $X(s), Y(s)$ of degree $n+q-1$, with $X(s)$ monic, which satisfy the equation

$$R_p Q_m X + Z_p Q_1 Y = A^*. \quad (6C.37)$$

$A^*(s)$ is a Hurwitz polynomial of degree $2(n+q)-1$ and contains all the common stable zeros of $Z_p Q_1, R_p Q_m$ if any. Dividing each side of (6C.37) with $A^*(s)$ and using it to operate on u_p , we obtain

$$\frac{R_p Q_m X}{A^*} u_p + \frac{Q_1 Y Z_p}{A^*} u_p = u_p. \quad (6C.38)$$

Using $Q_m u_p = Q_1 \bar{u}_p$ and $Z_p u_p = R_p y_p$ in (6C.38), we obtain

$$u_p = \frac{R_p X Q_1}{A^*} \bar{u}_p + \frac{Q_1 Y R_p}{A^*} y_p. \quad (6C.39)$$

The plant output satisfies

$$y_p = C^\top e_o + C^\top \hat{e} + y_m. \quad (6C.40)$$

Step 2. Establish e.s. for the homogeneous part of (6C.36) For each frozen time t , we have $\det(sI - A_c) = \det(sI - \hat{A} + \hat{B}\hat{K}_c) = A_c^*(s)$ (where the last equality is obtained using (6.63)), which implies that $A_c(t)$ is stable at each frozen time t . If $\hat{Z}_p(s, t)Q_1(s)$ and $\hat{R}_p(s, t)Q_m(s)$ are strongly coprime, i.e., (\hat{A}, \hat{B}) is strongly controllable at each time t , the controller gains \hat{K}_c may be calculated at each time t using Ackermann's formula [3], i.e.,

$$\hat{K}_c = [0, 0, \dots, 0, 1]G_c^{-1}A_c^*(\hat{A}),$$

where

$$G_c \triangleq [\hat{B}, \hat{A}\hat{B}, \dots, \hat{A}^{n+q-1}\hat{B}]$$

is the controllability matrix of the pair (\hat{A}, \hat{B}) . Because (\hat{A}, \hat{B}) is assumed to be strongly controllable and $\hat{A}, \hat{B} \in L_\infty$ due to $\theta_p \in L_\infty$, we have $\hat{K}_c \in L_\infty$. Now,

$$\dot{\hat{K}}_c = [0, 0, \dots, 0, 1] \left\{ -G_c^{-1} \dot{G}_c G_c^{-1} A_c^*(\hat{A}) + G_c^{-1} \frac{d}{dt} A_c^*(\hat{A}) \right\}.$$

Because $\theta_p \in L_\infty$ and $\dot{\theta}_p \in L_2$, it follows that $\|\dot{\hat{K}}_c(t)\| \in L_2$, which, in turn, implies that $\|\dot{A}_c(t)\| \in L_2$. From A_c being pointwise stable and $\|\dot{A}_c(t)\| \in L_2$, we have that $A_c(t)$ is a u.a.s. matrix by applying Theorem A.8.6. Because A_0 is a stable matrix the homogeneous part of (6C.36) is e.s.

Step 3. Use the properties of the $\mathcal{L}_{2\delta}$ norm and the B–G lemma to establish boundedness Applying Lemmas A.5.9, A.5.10 to (6C.36), (6C.39), (6C.40), we obtain

$$\begin{aligned} \|\hat{e}\| &\leq c \|C^\top e_o\|, \\ \|y_p\| &\leq c \|C^\top e_o\| + c \|\hat{e}\| + c \leq c \|C^\top e_o\| + c, \\ \|u_p\| &\leq c \|\hat{e}\| + c \|y_p\| \leq c \|C^\top e_o\| + c, \end{aligned} \quad (6C.41)$$

where $\|\cdot\|$ denotes the $L_{2\delta}$ norm for some $\delta > 0$.

We relate the term $C^\top e_o$ with the estimation error by using (6C.36) to express $C^\top e_o$ as

$$C^\top e_o = C^\top (sI - A_o)^{-1} (\tilde{\theta}_1 e_1 - \tilde{\theta}_2 \bar{u}_p). \quad (6C.42)$$

Since (C, A_o) is in the observer canonical form, i.e.,

$$C^\top (sI - A_o)^{-1} = \frac{\alpha_{n+q-1}(s)}{A_o^*(s)},$$

where $A_o^*(s) = \det(sI - A_o)$, we have

$$C^\top e_o = \sum_{i=0}^{\bar{n}} \frac{s^{\bar{n}-i}}{A_o^*} (\tilde{\theta}_{1i} e_1 - \tilde{\theta}_{2i} \bar{u}_p),$$

where $\bar{n} = n + q - 1$, $\tilde{\theta}_i = [\tilde{\theta}_{i1}, \tilde{\theta}_{i2}, \dots, \tilde{\theta}_{i\bar{m}}]^\top$, $i = 1, 2$. We apply Lemma A.11.1 (Swapping Lemma 1) to each term under the summation in (6C.43) to obtain

$$\frac{s^{\bar{n}-i}}{A_o^*} \tilde{\theta}_{1i} e_1 = \frac{\Lambda_p(s) \mathcal{Q}_1(s)}{A_o^*(s)} \left(\tilde{\theta}_{1i} \frac{s^{\bar{n}-i}}{\Lambda_p(s) \mathcal{Q}_1(s)} e_1 + W_{ci}(s) (W_{bi}(s) e_1) \tilde{\theta}_{1i} \right)$$

and

$$\frac{s^{\bar{n}-i}}{A_o^*} \tilde{\theta}_{2i} \bar{u}_p = \frac{\Lambda_p(s) \mathcal{Q}_1(s)}{A_o^*(s)} \left(\tilde{\theta}_{2i} \frac{s^{\bar{n}-i}}{\Lambda_p(s) \mathcal{Q}_1(s)} \bar{u}_p + W_{ci}(s) (W_{bi}(s) \bar{u}_p) \tilde{\theta}_{2i} \right),$$

where $W_{ci}, W_{bi}, i = 0, \dots, n + q - 1$ are transfer matrices defined in Lemma A.11.1 with $W(s) = \frac{s^{\bar{n}-i}}{\Lambda_p(s) \mathcal{Q}_1(s)}$. Therefore, $C^\top e_o$ can be expressed as

$$\begin{aligned} C^\top e_o &= \frac{\Lambda_p(s) \mathcal{Q}_1(s)}{A_o^*(s)} \sum_{i=0}^{\bar{n}} \left(\tilde{\theta}_{1i} \frac{s^{\bar{n}-i}}{\Lambda_p(s) \mathcal{Q}_1(s)} e_1 - \tilde{\theta}_{2i} \frac{s^{\bar{n}-i}}{\Lambda_p(s) \mathcal{Q}_1(s)} \bar{u}_p \right) + r_1 \\ &= \frac{\Lambda_p(s) \mathcal{Q}_1(s)}{A_o^*(s)} \left(\tilde{\theta}_1^\top \frac{\alpha_{n+q-1}(s)}{\Lambda_p(s) \mathcal{Q}_1(s)} e_1 - \tilde{\theta}_2^\top \frac{\alpha_{n+q-1}(s)}{\Lambda_p(s) \mathcal{Q}_1(s)} \bar{u}_p \right) + r_1, \end{aligned} \quad (6C.43)$$

where

$$r_1 \triangleq \frac{\Lambda_p(s) \mathcal{Q}_1(s)}{A_o^*(s)} \sum_{i=0}^{\bar{n}} W_{ci}(s) [(W_{bi}(s) e_1) \tilde{\theta}_{1i} - (W_{bi}(s) \bar{u}_p) \tilde{\theta}_{2i}].$$

From the definition of $\tilde{\theta}_1$, we have

$$\begin{aligned} \tilde{\theta}_1^\top \alpha_{n+q-1}(s) &= \theta_1^\top \alpha_{n+q-1}(s) - \theta_1^{*\top} \alpha_{n+q-1}(s) \\ &= \hat{R}_p(s, t) \mathcal{Q}_m(s) - s^{n+q} - R_p(s) \mathcal{Q}_m(s) + s^{n+q} \\ &= (\hat{R}_p(s, t) - R_p(s)) \mathcal{Q}_m(s) = \tilde{\theta}_a^\top \alpha_{n-1}(s) \mathcal{Q}_m(s), \end{aligned} \quad (6C.44)$$

where $\tilde{\theta}_a \triangleq \theta_a - \theta_a^*$ is the parameter error. Similarly,

$$\tilde{\theta}_2^\top \alpha_{n+q-1}(s) = \theta_b^\top \alpha_{n-1}(s) \mathcal{Q}_1(s), \quad (6C.45)$$

where $\tilde{\theta}_b \triangleq \theta_b - \theta_b^*$. Using (6C.44) and (6C.45) in (6C.43), we obtain

$$\begin{aligned} C^\top e_o &= \frac{\Lambda_p(s) \mathcal{Q}_1(s)}{A_o^*(s)} \left(\tilde{\theta}_a^\top \alpha_{n-1}(s) \frac{\mathcal{Q}_m(s)}{\mathcal{Q}_1(s)} \frac{1}{\Lambda_p(s)} e_1 - \tilde{\theta}_b^\top \alpha_{n-1}(s) \frac{1}{\Lambda_p(s)} \bar{u}_p \right) + r_1 \\ &= \frac{\Lambda_p(s) \mathcal{Q}_1(s)}{A_o^*(s)} \left(\tilde{\theta}_a^\top \alpha_{n-1}(s) \frac{\mathcal{Q}_m(s)}{\mathcal{Q}_1(s) \Lambda_p(s)} y_p - \tilde{\theta}_b^\top \alpha_{n-1}(s) \frac{\mathcal{Q}_m(s)}{\Lambda_p(s) \mathcal{Q}_1(s)} u_p \right) + r_1, \end{aligned} \quad (6C.46)$$

where the second equality is obtained using

$$Q_m(s)e_1 = Q_m(s)y_p, \quad Q_1(s)\bar{u}_p = Q_m(s)u_p.$$

Using the identities

$$\alpha_{n-1}(s)\frac{1}{\Lambda_p(s)}u_p = \phi_1, \quad \alpha_{n-1}(s)\frac{1}{\Lambda_p(s)}y_p = -\phi_2$$

and Lemma A.11.1, we obtain the following equalities:

$$\frac{Q_m(s)}{Q_1(s)}\tilde{\theta}_a^\top\phi_2 = -\tilde{\theta}_a^\top\alpha_{n-1}(s)\frac{Q_m(s)}{Q_1(s)\Lambda_p(s)}y_p + W_{cq}\left(W_{bq}(s)\phi_2^\top\right)\tilde{\theta}_a,$$

$$\frac{Q_m(s)}{Q_1(s)}\tilde{\theta}_b^\top\phi_1 = \tilde{\theta}_b^\top\alpha_{n-1}(s)\frac{Q_m(s)}{Q_1(s)\Lambda_p(s)}u_p + W_{cq}\left(W_{bq}(s)\phi_1^\top\right)\tilde{\theta}_b,$$

where W_{cq}, W_{bq} are as defined in Swapping Lemma A.11.1 with $W(s) = \frac{Q_m(s)}{Q_1(s)}$. Using the above equalities in (6C.46), we obtain

$$C^\top e_o = -\frac{\Lambda_p(s)Q_m(s)}{A_o^*(s)}\tilde{\theta}_p^\top\phi + r_2, \quad (6C.47)$$

where

$$r_2 \triangleq r_1 + \frac{\Lambda_p(s)Q_1(s)}{A_o^*(s)}\left(W_{cq}(s)\left(W_{bq}(s)\phi_1^\top\right)\tilde{\theta}_b + W_{cq}(s)\left(W_{bq}(s)\phi_2^\top\right)\tilde{\theta}_a\right).$$

The normalized estimation error satisfies the equation

$$\varepsilon m_s^2 = -\tilde{\theta}_p^\top\phi,$$

which can be used in (6C.47) to obtain

$$C^\top e_o = \frac{\Lambda_p(s)Q_m(s)}{A_o^*(s)}\varepsilon m_s^2 + r_2. \quad (6C.48)$$

Using the definition of the fictitious normalizing signal $m_f^2 \triangleq 1 + \|u_p\|^2 + \|y_p\|^2$ and Lemma A.5.2, we can show that $\phi/m_f, m_s/m_f \in L_\infty$ for some $\delta > 0$. Using these properties of m_f applying Lemma A.5.2 in (6C.48), we obtain

$$\|C^\top e_o\| \leq c\|\varepsilon m_s m_f\| + c\|\dot{\theta}_p m_f\|. \quad (6C.49)$$

From (6C.41) and in the definition of m_f , we have

$$m_f^2 = 1 + \|u_p\|^2 + \|y_p\|^2 \leq c\|C^\top e_o\|^2 + c,$$

which together with (6C.49) implies

$$m_f^2 \leq c \left\| \varepsilon m_s m_f \right\|^2 + \left\| \dot{\theta}_p m_f \right\|^2 + c$$

or

$$m_f^2 \leq c \left\| g m_f \right\|^2 + c,$$

where $g^2 \triangleq \varepsilon^2 m_s^2 + |\dot{\theta}_p|^2$ and $g \in L_2$ due to the properties of the adaptive law. If we now apply the B–G lemma to the above inequality we can show that $m_f \in L_\infty$. From $m_f \in L_\infty$ and the properties of the $L_{2\delta}$ norm we can establish boundedness for the rest of the signals.

Step 4. Convergence of the tracking error to zero Since all signals are bounded we can establish that $\frac{d}{dt}(\varepsilon m_s^2) \in L_\infty$, which, together with $\varepsilon m_s^2 \in L_2$, implies that $\varepsilon(t)m_s^2(t) \rightarrow 0$ and, therefore, $\dot{\theta}_p(t) \rightarrow 0$ as $t \rightarrow \infty$. From the expressions of r_1, r_2 we can conclude, using Corollary A.5.5, that $r_2 \in L_2$ and $r_2(t) \rightarrow 0$ as $t \rightarrow \infty$. Using $\varepsilon m_s^2, r_2 \in L_2$ and $r_2 \rightarrow 0$ as $t \rightarrow \infty$ in (6C.48) and applying Corollary A.5.5, we have that $|C^\top e_o| \in L_2$ and $|C^\top e_o(t)| \rightarrow 0$ as $t \rightarrow \infty$.

Consider the first equation in (6C.36), i.e.,

$$\hat{\dot{e}} = A_c(t)\hat{e} + K_o C^\top e_o.$$

Since $A_c(t)$ is u.a.s. and $|C^\top e_o| \in L_2$, $|C^\top e_o| \rightarrow 0$ as $t \rightarrow \infty$ we can conclude that $\hat{e} \rightarrow 0$ as $t \rightarrow \infty$. Since $e_1 \triangleq y_p - y_m = C^\top e_o + C^\top \hat{e}$ (see (6C.39)), it follows that $e_1(t) \rightarrow 0$ as $t \rightarrow \infty$ and the proof is complete. \square

6.7.3 Robust APPC: Polynomial Approach

Proof of Theorem 6.7.2.

Step 1. Express u_p, y_p in terms of the estimation error Following exactly the same steps as in the proof of Theorem 6.3.2 for the ideal case, we can show that the input u_p and output y_p satisfy the same equations as in (6C.10), expressed as

$$\begin{aligned} \dot{x} &= A(t)x + b_1(t)\varepsilon m_s^2 + b_2 \bar{y}_m, \\ y_p &= C_1^\top x + d_1 \varepsilon m_s^2 + d_2 \bar{y}_m, \\ u_p &= C_2^\top x + d_3 \varepsilon m_s^2 + d_4 \bar{y}_m, \end{aligned} \tag{6C.50}$$

where $x, A(t), b_1(t), b_2$, and $C_i, i=1,2; d_k, k=1,2,3,4$ are as defined in (6C.10) and $\bar{y}_m = \hat{P} \frac{1}{\Lambda(s)} y_m$. The modeling error terms due to $\Delta_m(s), d_u$ do not appear explicitly in (6C.50). Due to the unmodeled dynamics $\varepsilon, \varepsilon m_s, \dot{A}(t)$ no longer belong to L_2 as in the ideal case, but belong to $S(\frac{\eta^2}{m^2} + f_0)$. Because $\frac{\eta^2}{m^2} \leq \Delta_2^2 + \frac{d_0^2}{m^2}$, we have

$$\varepsilon, \varepsilon m_s, \dot{A}(t) \in S\left(\Delta_2^2 + \frac{d_0^2}{m^2} + f_0\right).$$

Step 2. Establish the u.a.s. property of $A(t)$ As in the ideal case, the APPC law guarantees that $\det(sI - A(t)) = A^*(s)$ for each time t where $A^*(s)$ is Hurwitz. If we apply Theorem A.8.6(ii) to the homogeneous part of (6C.50), we can establish that $A(t)$ is u.a.s. which is equivalent to e.s., provided

$$c \left(f_0 + \Delta_2^2 + \frac{1}{T} \int_t^{t+T} \frac{d_0^2}{m_s^2} d\tau \right) < \mu^* \quad (6C.51)$$

$\forall t \geq 0$, any $T \geq 0$, and some $\mu^* > 0$, where $c \geq 0$ is a finite constant. Because $m_s^2 > 0$, condition (6C.51) may not be satisfied for small Δ_2, f_0 unless d_0 , the upper bound for the input disturbance, is zero or sufficiently small. As in the MRAC case, we can deal with the disturbance in two different ways. One way is to modify $m_s^2 = 1 + n_d$ to $m_s^2 = 1 + n_d + \beta_0$, where β_0 is chosen to be large enough so that $c \frac{d_0^2}{m^2} \leq c \frac{d_0^2}{\beta_0} < \frac{\mu^*}{2}$, say, so that for

$$c(f_0 + \Delta_2^2) < \frac{\mu^*}{2}$$

condition (6C.51) is always satisfied and $A(t)$ is e.s. The other way is to keep the same m_s^2 and establish that when m_s^2 grows large over an interval of time $I_1 = [t_1, t_2]$, say, the state-transition matrix $\Phi(t, \tau)$ of $A(t)$ satisfies $\Phi(t, \tau) \leq k_1 e^{-k_2(t-\tau)} \quad \forall t \geq \tau$ and $t, \tau \in I_1$. This property of $A(t)$ (when m_s^2 grows large) is used in Step 3 to contradict the hypothesis that m_s^2 could grow unbounded and conclude boundedness. We proceed as follows.

Let us start by assuming that m_s^2 grows unbounded. Because all the elements of the state x are the outputs of strictly proper transfer functions with the same poles as the roots of $\Lambda(s)$ and inputs u_p, y_p and the roots of $\Lambda(s)$ are located in $\text{Re}[s] < -\delta_0/2$, it follows from Lemma A.5.2 that $\frac{x}{m_s} \in L_\infty$. Because $\bar{y}_m, \varepsilon m_s \in L_\infty$, it follows from (6C.51) that $\frac{y_p}{m_s}, \frac{u_p}{m_s} \in L_\infty$. Because u_p^2, y_p^2 are bounded from above by m_s^2 , it follows from the equation for m_s^2 that m_s^2 cannot grow faster than an exponential, i.e., $m_s^2(t) \leq e^{k_1(t-t_0)} m_s^2(t_0) \quad \forall t \geq t_0 \geq 0$ for some $k > 0$. Because $m_s^2(t)$ is assumed to grow unbounded, we can find a $t_0 > \bar{\alpha} > 0$ for any arbitrary constant $\bar{\alpha} > t_2 - t_0$ such that $m_s^2(t_2) > \bar{\alpha} e^{k_1 \bar{\alpha}}$. We have

$$\bar{\alpha} e^{k_1 \bar{\alpha}} < m_s^2(t_2) \leq e^{k_1(t_2-t_0)} m_s^2(t_0),$$

which implies that

$$\ln m_s^2(t_0) > \ln \bar{\alpha} + k_1[\bar{\alpha} - (t_2 - t_0)].$$

Because $\bar{\alpha} > t_2 - t_0$ and $t_0 \in (t_2 - \bar{\alpha}, t_2)$, it follows that

$$m_s^2(t_0) > \bar{\alpha} \quad \forall t_0 \in (t_2 - \bar{\alpha}, t_2).$$

Let $t_1 = \sup_{\tau \leq t_2} \arg(m_s^2(\tau) = \bar{\alpha})$. Then, $m_s^2(t_1) = \bar{\alpha}$ and $m_s^2(t) \geq \bar{\alpha} \forall t \in [t_1, t_2)$, where $t_1 \leq t_2 - \bar{\alpha}$, i.e., $t_2 - t_1 \geq \bar{\alpha}$. Let us now consider the behavior of the homogeneous part of (6C.50), i.e.,

$$\dot{Y} = A(t)Y \quad (6C.52)$$

over the interval $I_1 \triangleq [t_1, t_2)$ for which $m_s^2(t) \geq \bar{\alpha}$ and $t_2 - t_1 \geq \bar{\alpha}$, where $\bar{\alpha} > 0$ is an arbitrary constant. Because $\det(sI - A(t)) = A^*(s)$, i.e., $A(t)$ is a pointwise stable matrix, the Lyapunov equation

$$A^\top(t)P(t) + P(t)A(t) = -I \quad (6C.53)$$

has the solution $P(t) = P^\top(t) > 0$ for each $t \in I_1$. We consider the Lyapunov function

$$V(t) = Y^\top(t)P(t)Y(t);$$

then

$$\dot{V} = -Y^\top Y + Y^\top \dot{P}Y \leq -Y^\top Y + \dot{P}(t)Y^\top Y.$$

Using (6C.53) and the boundedness of P, A , we can establish that $\|\dot{P}(t)\| \leq c \|\dot{A}(t)\|$. Because $\lambda_1 Y^\top Y \leq V \leq \lambda_2 Y^\top Y$ for some $0 < \lambda_1 < \lambda_2$, it follows that

$$\dot{V} \leq -(\lambda_2^{-1} - c\lambda_1^{-1}\dot{A}(t))V;$$

i.e.,

$$V(t) \leq e^{-\int_\tau^t (\lambda_2^{-1} - c\lambda_1^{-1}\dot{A}(s))ds} V(\tau)$$

$\forall t \geq \tau \geq 0$. For the interval $I_1 = [t_1, t_2)$, we have $m_s^2(t) \geq \bar{\alpha}$ and, therefore,

$$\int_\tau^t \|\dot{A}(\tau)\| d\tau \leq c \left(\Delta_2^2 + f_0 + \frac{d_0^2}{\bar{\alpha}} \right) (t - \tau) + c.$$

Therefore,

$$V(t) \leq e^{-\lambda_0(t-\tau)} V(\tau) \quad \forall t, \tau \in [t_1, t_2) \quad (6C.54)$$

and $t \geq \tau$, provided

$$c \left(f_0 + \Delta_2^2 + \frac{d_0^2}{\bar{\alpha}} \right) < \lambda_0, \quad (6C.55)$$

where $\lambda_0 = \frac{\lambda_2^{-1}}{2}$. From (6C.54) we have

$$\lambda_1 Y^\top(t)Y(t) \leq Y^\top(t)PY(t) \leq e^{-\lambda_0(t-\tau)} \lambda_2 Y^\top(\tau)Y(\tau),$$

which implies that

$$|Y(t)| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} e^{-\lambda_0(t-\tau)} |Y(\tau)| \quad \forall t, \tau \in [t_1, t_2]$$

which, in turn, implies that the transition matrix $\Phi(t, \tau)$ of (6C.52) satisfies

$$\|\Phi(t, \tau)\| \leq \beta_0 e^{-\alpha_0(t-\tau)} \quad \forall t, \tau \in [t_1, t_2], \quad (6C.56)$$

where $\beta_0 = \sqrt{\frac{\lambda_2}{\lambda_1}}, \alpha_0 = \frac{\lambda_0}{2}$. Condition (6C.55) can be satisfied by choosing $\bar{\alpha}$ large enough and by requiring Δ_2, f_0 to be smaller than some constant, i.e., $c(f_0 + \Delta_2^2) < \frac{\lambda_2^{-1}}{4}$, say. In the next step we use (6C.56) and continue our argument over the interval I_1 in order to establish boundedness by contradiction.

Step 3. Boundedness using the B–G lemma and contradiction We apply Lemma A.5.10 to (6C.50) for $t \in [t_1, t_2)$. We have

$$\|x_{t,t_1}\| \leq ce^{-\delta/2(t-t_1)} |x(t_1)| + c\|(\varepsilon m_s^2)_{t,t_1}\| + c,$$

where $\|(\cdot)_{t,t_1}\|$ denotes the $L_{2\delta}$ norm $\|(\cdot)_{t,t_1}\|_{2\delta}$ defined over the interval $[t_1, t)$, for any $0 < \delta < \delta_1 < 2\alpha_0$. Because $\frac{x}{m_s} \in L_\infty$ it follows that

$$\|x_{t,t_1}\| \leq ce^{-\delta/2(t-t_1)} m_s(t_1) + c\|(\varepsilon m_s^2)_{t,t_1}\| + c.$$

Because $\|y_{pt,t_1}\|, \|u_{pt,t_1}\| \leq c\|x_{t,t_1}\| + c\|(\varepsilon m_s^2)_{t,t_1}\| + c$, it follows that

$$\|y_{pt,t_1}\|, \|u_{pt,t_1}\| \leq ce^{-\delta/2(t-t_1)} m_s(t_1) + c\|(\varepsilon m_s^2)_{t,t_1}\| + c.$$

Now $m_s^2(t) = 1 + n_d(t)$ and

$$n_d(t) = e^{-\delta_0(t-t_1)} n_d(t_1) + \|y_{pt,t_1}\|_{2\delta_0}^2 + \|u_{pt,t_1}\|_{2\delta_0}^2.$$

Because $\|(\cdot)_{t,t_1}\|_{2\delta_0} \leq \|(\cdot)_{t,t_1}\|$ for $\delta \leq \delta_0$, it follows that

$$m_s^2(t) = 1 + n_d(t) \leq 1 + e^{-\delta_0(t-t_1)} m_s^2(t_1) + \|y_{pt,t_1}\|^2 + \|u_{pt,t_1}\|^2 \quad \forall t \geq t_1.$$

Substituting for the bound for $\|y_{pt,t_1}\|, \|u_{pt,t_1}\|$, we obtain

$$m_s^2(t) \leq ce^{-\delta(t-t_1)} m_s^2(t_1) + c\|(\varepsilon m_s^2)_{t,t_1}\|^2 + c \quad \forall t \geq t_1 \geq 0$$

or

$$m_s^2(t) \leq c + ce^{-\delta(t-t_1)} m_s^2(t_1) + c \int_{t_1}^t e^{-\delta(t-\tau)} \varepsilon^2 m_s^2(\tau) d\tau. \quad (6C.57)$$

Applying B–G lemma 3 (Lemma A.6.3),

$$m_s^2(t) \leq c(1 + m_s^2(t_1)) e^{-\delta(t-t_1)} e^{c \int_{t_1}^t \varepsilon^2 m_s^2 d\tau} + c\delta \int_{t_1}^t e^{-\delta(t-s)} e^{c \int_s^t \varepsilon^2 m_s^2 d\tau} ds \quad \forall t \geq t_1.$$

For $t, s \in [t_1, t_2)$ we have

$$c \int_s^t \varepsilon^2 m_s^2 d\tau \leq c \left(\Delta_2^2 + f_0 + \frac{d_0^2}{\bar{\alpha}} \right) (t-s) + c.$$

By choosing $\bar{\alpha}$ large enough so that $c \frac{d_0^2}{\bar{\alpha}} < \frac{\delta}{4}$ and by requiring

$$c(\Delta_2^2 + f_0) < \frac{\delta}{4},$$

we have

$$m_s^2(t_2) \leq c(1 + m_s^2(t_1)) e^{-\frac{\delta}{2}(t_2-t_1)} + c\delta \int_{t_1}^{t_2} e^{-\frac{\delta}{2}(t_2-s)} ds \leq c(1 + m_s^2(t_1)) e^{-\frac{\delta}{2}(t_2-t_1)} + c.$$

Since $t_2 - t_1 \geq \bar{\alpha}$, $m_s^2(t_1) = \bar{\alpha}$, and $m_s^2(t_2) > \bar{\alpha}$, we have

$$\bar{\alpha} < m_s^2(t_2) \leq c(1 + \bar{\alpha}) e^{-\frac{\delta \bar{\alpha}}{2}} + c.$$

Therefore, we can choose $\bar{\alpha}$ large enough so that $m_s^2(t_2) < \bar{\alpha}$, which contradicts the hypothesis that $m_s^2(t_2) > \bar{\alpha}$. Therefore, $m_s \in L_\infty$, which implies that $x, u_p, y_p \in L_\infty$.

The condition for robust stability is, therefore,

$$c(f_0 + \Delta_2^2) < \min \left\{ \frac{\lambda_2^{-1}}{2}, \frac{\delta}{4} \right\} \triangleq \delta^*$$

for some finite constant $c > 0$.

Step 4. Establish bounds for the tracking error A bound for the tracking error e_1 is obtained by expressing e_1 in terms of signals that are guaranteed by the adaptive law to be of the order of the modeling error in the m.s.s. The tracking error equation has exactly the same form as in the ideal case and is given by

$$e_1 = \frac{\Lambda(s) s^{n-1} Q_m(s)}{A^*(s)} \varepsilon m_s^2 + \frac{\Lambda(s) \alpha_{n-2}^\top(s)}{A^*(s)} v_0,$$

where v_0 is the output of proper stable transfer functions whose inputs are elements of $\dot{\theta}_p$ multiplied by bounded signals. Because $\dot{\theta}_p, \varepsilon m_s^2 \in S(\frac{n^2}{m_s^2} + f_0)$ and $\frac{n^2}{m_s^2} \leq c(\Delta_2^2 + d_0^2)$, due to $m_s \in L_\infty$, it follows from Corollary A.5.8 that $e_1 \in S(\Delta_2^2 + d_0^2 + f_0)$ and the proof is complete. \square

6.9 Examples Using the Adaptive Control Toolbox

The PPC and APPC algorithms presented in Chapter 6 can be implemented using a set of MATLAB functions and the Simulink block **Adaptive Controller** provided in the Adaptive Control Toolbox. For implementation of the APPC algorithms, these tools need to be used together with the PI functions or blocks introduced in Chapter 3. In this section, we demonstrate the use of the Adaptive Control Toolbox in various PPC and APPC problems via a number of simulation examples.

6.9.1 PPC: Known Parameters

The Adaptive Control Toolbox provides an easy way of designing and simulating a wide range of PPC schemes. For the polynomial approach, the MATLAB function **ppcpoly** can be used to solve (6.31) for a given plant of the form (6.28) and design polynomials $A^*(s)$ and $\Lambda(s)$. Denoting the coefficient vectors of $Z(s), R(s), A^*(s), \Lambda(s), Q_m(s)$ as $Z, R, As, Lambda, Qm$, respectively,

```
[Ku, Ky, RETYPE] = ppcpoly(Z, R, As, Lambda, Qm)
```

returns the coefficient vectors of K_u and K_y . RETYPE is returned as 0 if the equation solving process is successful and as -1 if it fails.

Consider the SPM

$$z = \theta^{*T} \phi \quad (6C.58)$$

for the plant

$$y = \frac{Z(s)}{R(s)} u = \frac{b_m s^m + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} u,$$

where

$$\begin{aligned} \theta^* &= [b_m \quad \dots \quad b_0 \quad a_{n-1} \quad \dots \quad a_0]^T, \\ z &= \frac{s^n}{\Lambda_p(s)} y, \\ \phi &= \left[\frac{s^m}{\Lambda_p(s)} u \quad \dots \quad \frac{1}{\Lambda_p(s)} u \quad -\frac{s^{n-1}}{\Lambda_p(s)} y \quad \dots \quad -\frac{1}{\Lambda_p(s)} y \right]^T, \end{aligned}$$

and $\Lambda_p(s)$ is a monic Hurwitz design polynomial of order n . This SPM is also used in parameterization for adaptive PPC, in section 6.9.2.

The MATLAB function `uppcpoly` or the Simulink block **Adaptive Controller** with appropriate options and parameters can be used to generate the signals ϕ and z of the SPM (6C.58), and the control signal u , as demonstrated in Examples 6.9.1 and 6.9.2 below.

Example 6.9.1 Consider the plant $y = \frac{2}{s-1}u$. We want to choose u so that the closed-loop poles are equal to the roots of $A^*(s) = (s+1)^2$ and y tracks the reference signal $y_m = 1$. The reference input, y_m , satisfies $Q_m(s)y_m = 0$ for $Q_m(s) = s$. The control signal can be generated as

$$u = \frac{K_u(s)}{\Lambda(s)}u + \frac{K_y(s)}{\Lambda(s)}(y - y_m), \quad \Lambda(s) = s + 0.5.$$

$K_u(s)$ and $K_y(s)$ can be calculated using the following code:

```
Z = 2;
R = [1 -1];
As = [1 2 1];
Lambda = [1 0.5];
Qm = [1 0];
[Ku, Ky, RETYPE] = uppcpoly(Z, R, As, Lambda, Qm)
```

The result is $K_u(s) = 0.5$, $K_y(s) = -1.5s - 0.5$. ■

Example 6.9.2 In Example 6.9.1, we have calculated the control polynomials as $K_u(s) = 0.5$ and $K_y(s) = -1.5s - 0.5$. Now we can generate the linear model and control signals using the following code (added to the code of Example 6.9.1):

```
Lambdap = Lambda;
dt = 0.005; % Time increment for simulation
(sec).
t = [0:dt:10]; % Process time (sec).
lt = length(t);
ym = ones(1,lt); % Reference signal.
% State initialization for plant:
[nstate, xp0] = ufilts('init', Zp, Rp, 0);
xp(:,1) = xp0;
% State initialization for linear parametric model:
[nstate xl0] = uppcpoly('init', [1 0], Lambda, Lambdap, Qm);
xl(:,1) = xl0;
% Signal initialization:
y(1) = 0;
% Process
for k = 1:lt,
    u(k) = uppcpoly('control', xl(:,k), [y(k) ym(k)], [Ku
Ky], ... Lambda, Lambdap, Qm);
    [z(k), phi(:,k)] = uppcpoly('output', xl(:,k), [u(k)
y(k)], [1 0], ... Lambda, Lambdap, Qm);
    dxl = uppcpoly('state', xl(:,k), [u(k) y(k) ym(k)], [1 0],
... Lambda, Lambdap, Qm);
    xl(:,k+1) = xl(:,k) + dt*dxl;
    dxp = ufilts('state', xp(:,k), u(k), Zp, Rp);
    xp(:,k+1) = xp(:,k) + dt*dxp;
```

```
y(k+1) = ufilt('output', xp(:, k+1), Zp, Rp);  
end  
% Outputs:  
y = y(1:1t);
```

The results are shown in Figures 6C.1 and 6C.2. Another way of simulating the above scheme is to use the Simulink block **Adaptive Controller**. We can use the Simulink scheme shown in Figure 6C.3, and choose the appropriate options and enter the appropriate parameters in the menus of the **Adaptive Controller** block.

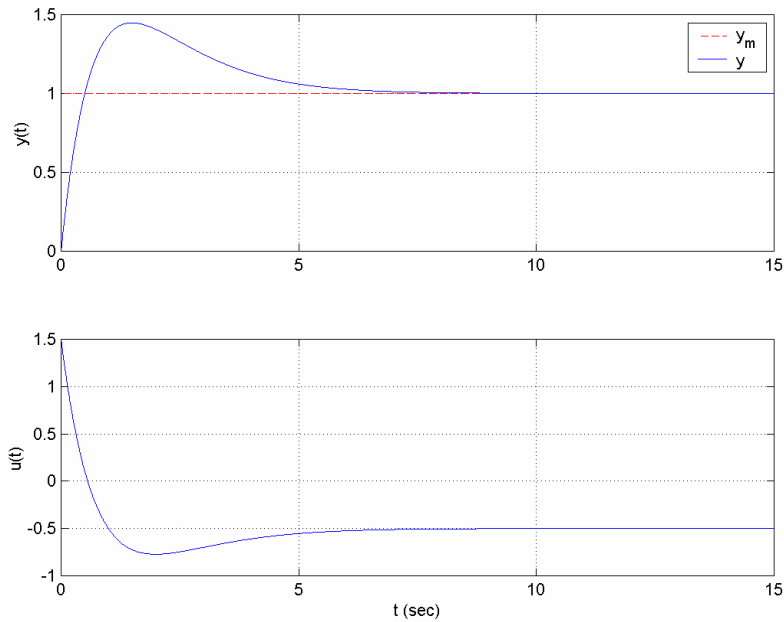


Figure 6C.1 Output and control signals of Example 6.9.2.

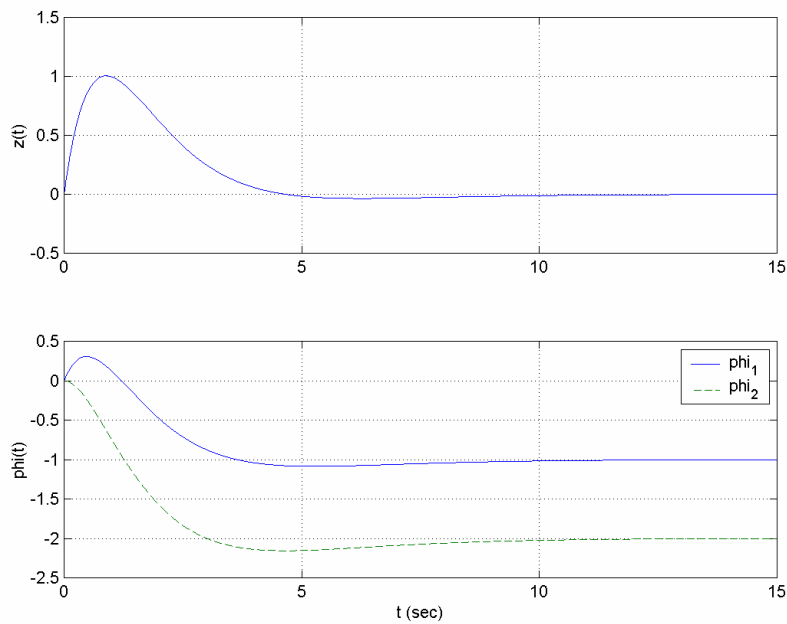


Figure 6C.2 SPM signals of Example 6.9.2.

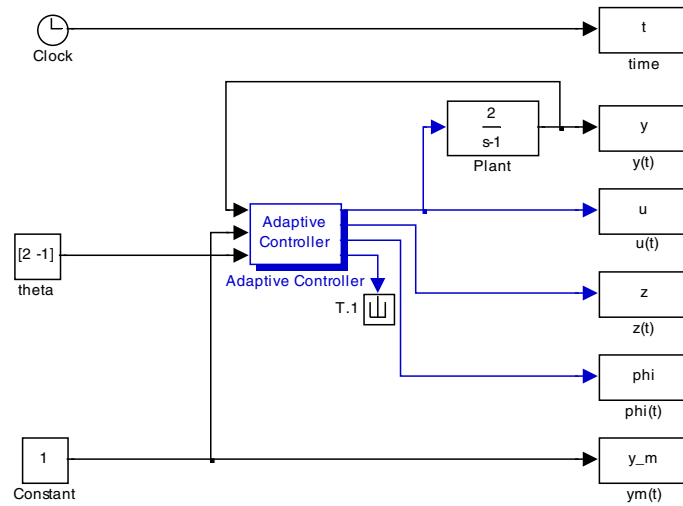


Figure 6C.3 Simulink implementation of the PPC Scheme in Example 6.9.2.

For the PPC scheme based on the state-feedback approach, where the state vector e of the state-space representation (6.52) for the plant (6.50), i.e.,

$$\dot{e}(t) = Ae(t) + B\bar{u}_p(t), \quad e_1 = C^T e,$$

is available for measurement, the MATLAB function **ppcssv** can be used to compute the state-feedback gain K_c of the PPC law

$$u = -K_c e$$

for a given design polynomial $A^*(s)$. Denoting the coefficient vectors of $Z(s)$, $R(s)$, $A^*(s)$ by Z , R , As , respectively,

$$[Kc, RETYPE] = \text{ppcssv}(Z, R, As)$$

returns the feedback gain K_c . $RETYPE$ is returned as 0 if the equation solving process is successful and as -1 if it fails.

If the state vector of (6.52) is not available for measurement, then the value of the Luenberger observer gain K_o in (6.55), which is required in estimation of the state vector e , can be computed using the MATLAB function **stateest**. Denoting the desired characteristic polynomial of $A_c - K_o C_c^T$ by $A_o^*(s)$, and the coefficient vector of $A_o^*(s)$ by Aos ,

$$[K_o, RETYPE] = \text{stateest}(C, R, Aos)$$

returns the observer gain K_o . $RETYPE$ is returned as 0 if the equation solving process is successful and as -1 if it fails.

The MATLAB function **uppcrsf** or the Simulink block **Adaptive Controller** with appropriate options and parameters can be used to generate the signals ϕ and z of the SPM (6C.58), and the control signal u , as demonstrated in Example 6.9.4. ■

Example 6.9.3 Consider the plant $y = \frac{2}{s-1}u$. We like to choose u so that the closed-loop poles are equal to the roots of $A^*(s) = (s+1)^2$ and y tracks the reference signal $y_m = 1$. This problem can be converted to a regulation problem by considering the tracking error equation

$$e_1 = y - y_m = \frac{2(s+1)}{(s-1)s} \bar{u}, \quad \bar{u} = \frac{s}{s+1} u,$$

where we want e_1 to converge to 0. Assuming that the states of the canonical representation

$$\dot{e} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} e + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \bar{u}, \quad \varepsilon = [1 \ 0] e$$

are available for measurement, we can design a state-variable controller

$$u = \frac{s+1}{s} \bar{u}, \quad \bar{u} = Ke.$$

The value of K can be calculated as follows:

$$R = [1 \ -1 \ 0];$$

$$Z = [2 \ 2];$$

$$As = [1 \ 2 \ 1];$$

$$[K, RETYPE] = \text{ppcssv}(Z, R, As)$$

The result is $K = [-1 \quad -0.5]$.

If the state vector e is not available for measurement, we use a state estimator. Let us design a state estimator with gain vector L so that $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - L[1 \quad 0]$ has the characteristic polynomial $A_o^* = (s+2)^2$. The following code can be used for this purpose:

```
R = [1 -1 0];
C = [1 0];
Aos = [1 4 4];
[L, RETYPE] = stateest(C, R, Aos)
```

The calculated observer gain is $L = [5 \quad 4]^T$. ■

Example 6.9.4 In Example 6.9.3, we have calculated the feedback gain as $K = [-1 \quad -0.5]$. Now we can generate the linear model and control signals using the following code:

```
Qm = [1 0];
Z = 2;
R = [1 -1];
Q1 = [1 1];
As = [1 2 1];
ba = [Z R(2)];
[num den] = uppcrsf('augment', [1 0], Q1, Qm, ba);
[K, RETYPE] = ppccssv(num, den, As)

Lambdap = Q1;
Aos = [1 4 4];

dt = 0.005; % Time increment for simulation (sec).
t = [0:dt:10]; % Process time (sec).
lt = length(t);
ym = ones(1,lt); % Reference signal.
% State initialization for plant:
[nstate, xp0] = ufilt('init', Z, R, 0);
xp(:,1) = xp0;
% State initialization for state space and linear parametric model:
[nstate xl0] = uppcrsf('init', [1 0], Aos, Q1, Qm, Lambdap);
xl(:,1) = xl0;
% Signal initialization:
y(1) = 0;
% Process
for k = 1:lt,
    [u(k) ubar(k)] =
uppcrsf('control', xl(:,k), K, Q1, Qm, Lambdap);
    [z(k), phi(:,k)] = uppcrsf('output', xl(:,k), [u(k)
y(k)], [1 0], ...Aos, Q1, Qm, Lambdap);
    dxl = uppcrsf('state', xl(:,k), [u(k) ubar(k) y(k)
ym(k)], [1 0], ...
Aos, Q1, Qm, Lambdap, ba);
    xl(:,k+1) = xl(:,k) + dt*dxl;
    dxp = ufilt('state', xp(:,k), u(k), Z, R);
```

```

xp(:,k+1)=xp(:,k) + dt*dxp;
y(k+1) = ufilt('output',xp(:,k+1),Z,R);
end
% Outputs:
y = y(1:1t);

```

The results are plotted in Figures 6C.4 and 6C.5. Another way of simulating the above schemes is to use the Simulink block **Adaptive Controller**. One can use a scheme similar to the one shown in Figure 6C.3, and choose the appropriate options and enter the appropriate parameters in the menus of the **Adaptive Controller** Simulink block. ■

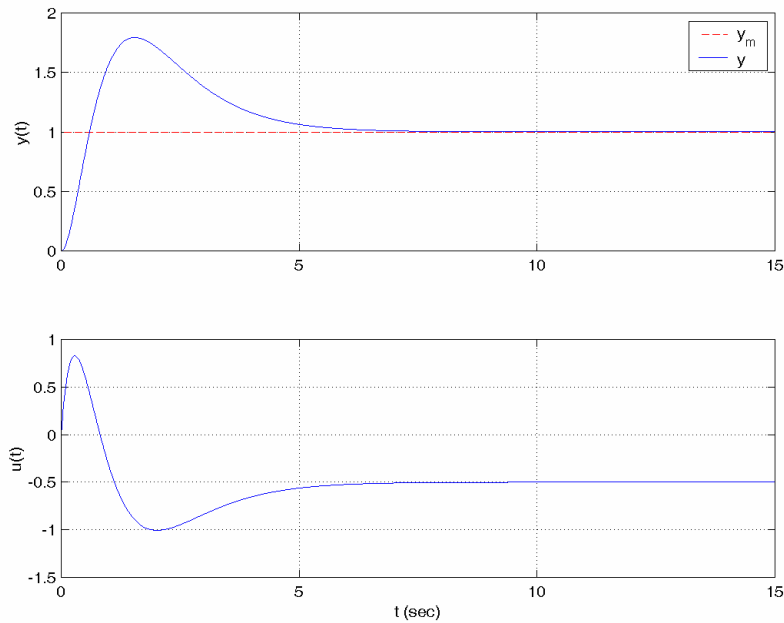


Figure 6C.4 Output and control signals of Example 6.9.4.

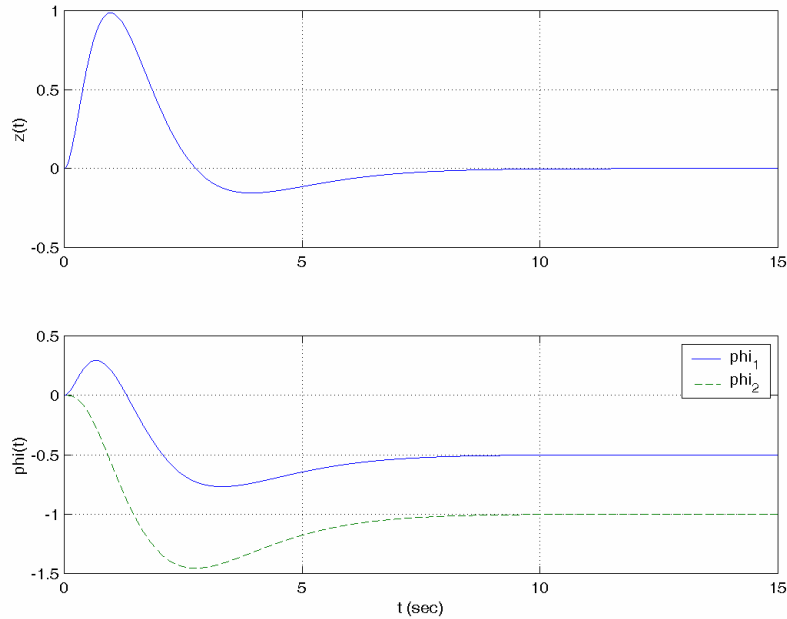


Figure 6C.5 Linear parametric model signals of Example 6.9.4.

For the linear quadratic (LQ) control, the MATLAB function `ppcclq` can be used to compute the state-feedback gain K for the plant (6.50) with the state-space representation (6.71) and a given penalty coefficient λ . Denoting λ and the coefficient vectors of $Z(s), R(s)$ by `lambda` and `Z, R`, respectively,

```
[K, RETYPE] = ppcclq(Z, R, lambda)
```

solves the Riccati equation (6.73) and returns the feedback gain K . `RETYPE` is returned as 0 if the equation solving process is successful and as -1 if it fails.

The MATLAB function `uppcrsf` or the Simulink block **Adaptive Controller** with appropriate options and parameters can be used to generate the signals ϕ and z of the SPM (6C.58), and the control signal u , as in the case of PPC, using a state-feedback approach.

Example 6.9.5 Consider the plant $y = \frac{2}{s-1}u$ of Example 6.9.3. We want to choose u so that the closed-loop plant is stable and y tracks the reference signal $y_m = 1$. As before, this problem can be converted to a regulation problem by considering the tracking error equation

$$e_1 = y - y_m = \frac{2(s+1)}{(s-1)s}\bar{u}, \quad \bar{u} = \frac{s}{s+1}u,$$

where we want e_1 to converge to 0. Assuming that the states of the canonical representation (or the states successfully estimated; see Example 6.9.3)

$$\dot{\varepsilon} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \varepsilon + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \bar{u}, \quad \varepsilon = \begin{bmatrix} 1 & 0 \end{bmatrix} e,$$

are available for measurement, we can design the LQ controller

$$u = \frac{s+1}{s}\bar{u}, \quad \bar{u} = Ke,$$

which minimizes the cost function $J = \int_0^\infty (e^2 + 0.2\bar{u}^2)dt$. The corresponding feedback gain K can be calculated as follows:

```
R = [1 -1 0];
Z = [2 2];
[K, RETYPE] = ppcc1q(Z, R, 0.2)
```

The result is $K = [-2.7361 \quad -0.5]$. If we apply this gain in Example 6.9.4 instead of $[-1 \quad -0.5]$, we obtain the results shown in Figures 6C.6 and 6C.7. ■

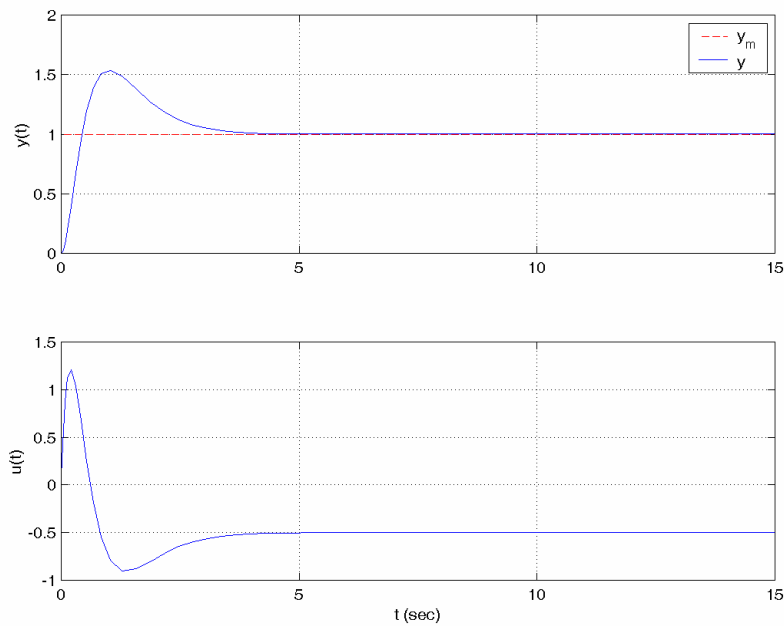


Figure 6C.6 Output and control signals of Example 6.9.5.

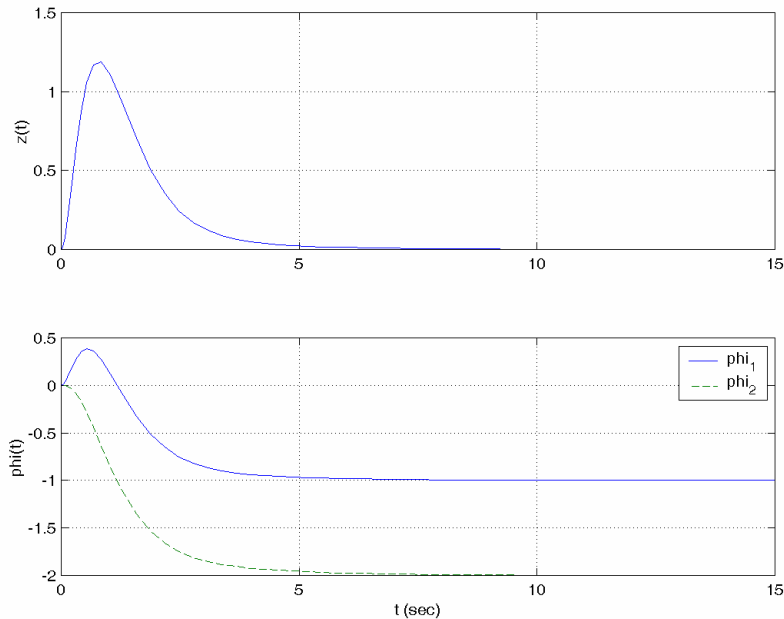


Figure 6C.7 Linear parametric model signals of Example 6.9.5.

6.9.2 APPC

An APPC scheme, in general, is composed of a PPC law that can be obtained using one of the three approaches mentioned in section 6.9.1 and a PI algorithm to estimate the unknown plant parameters, which can be chosen from a wide class of algorithms presented in Chapter 3.

The Adaptive Control Toolbox functions `ppcpoly`, `ppcssv`, `ppcclq` can be used to generate the control signals, as demonstrated in section 6.9.1, with unknown plant parameters. The functions `uppcpoly` and `uppcrsf` or the Simulink block **Adaptive Controller** with appropriate options and parameters incorporated with parameter identification functions (or the **Parameter Estimator** block) can be used to generate the signals ϕ and z of the linear model (6C.58), and the control signal u .

Example 6.9.6 Consider the plant of Example 6.9.2. Assume that the plant parameters are unknown. We want to apply the same control algorithm. We can use the linear model $z = \theta^{*T} \phi$, where

$$z = \frac{s^2}{(s+1)^2} y,$$

$$\theta_p^* = [2 \quad 2 \quad -1 \quad 0]^T,$$

$$\phi = \left[\frac{s}{(s+1)^2} u \quad \frac{1}{(s+1)^2} u \quad -\frac{s}{(s+1)^2} y \quad -\frac{s}{(s+1)^2} y \right]^T,$$

to estimate the unknown plant parameters. The Simulink blocks **Parameter Estimator** and **Adaptive Controller** can be used for simulation. Selecting the PI algorithm to be LS with forgetting factor $\beta = 1$ and $P_0 = 10 I$, and initializing θ

as $[1 \ 1 \ 1]^T$, the results plotted in Figures 6C.8 and 6C.9 demonstrate that both the estimation and tracking objectives are successful. ■

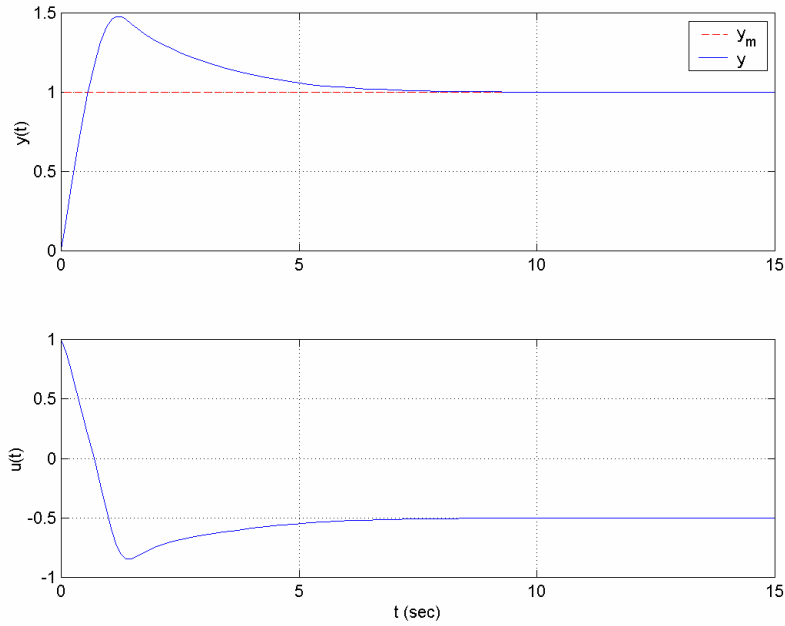


Figure 6C.8 Output and control signals of Example 6.9.6.

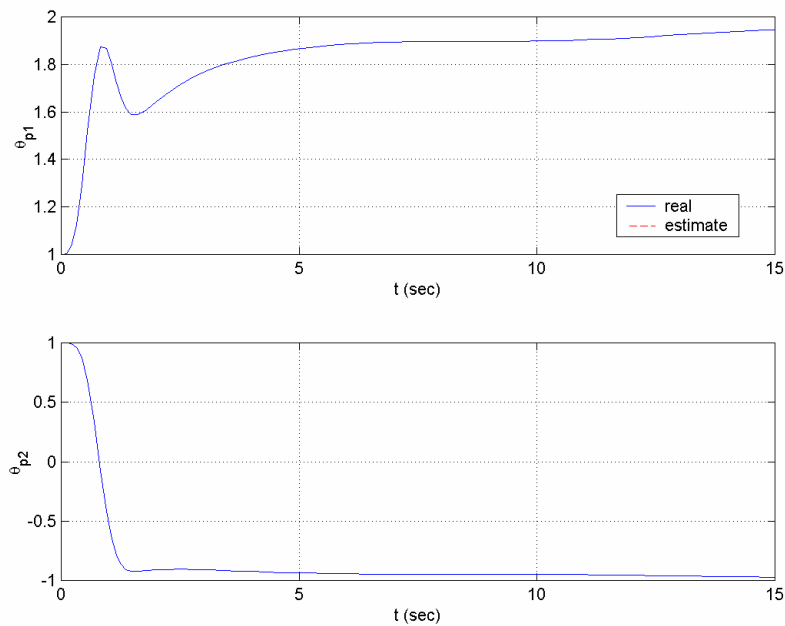


Figure 6C.9 SPM parameter estimate $\theta = [\hat{b}_1 \ \hat{a}_0]^T$ of Example 6.9.6.

Bibliography

- [1] T. KAILATH, *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980.
- [2] P. A. IOANNOU AND J. SUN, *Robust Adaptive Control*, Prentice-Hall, Englewood Cliffs, NJ, 1996; also available at

http://www-rcf.usc.edu/~ioannou/Robust_Adaptive_Control.htm.
- [3] B. D. O. ANDERSON, *Exponential stability of linear equations arising in adaptive identification*, IEEE Trans. Automat. Control, 22 (1977), pp. 83–88.