Preface

1 Objectives and Scope of the Book

The objective of this book is to give a comprehensive presentation of mathematical constructions and tools that can be used to study problems where the modeling, optimization, or control variable is no longer a set of parameters or functions but the shape or the structure of a geometric object. In that context, a good analytical framework and good modeling techniques must be able to handle the occurrence of singular behaviors whenever they are compatible with the mechanics or the physics of the problems at hand. In some optimization problems, the natural intuitive notion of a geometric domain undergoes mutations into relaxed entities such as microstructures. So the objects under consideration need not be smooth open domains, or even sets, as long as they still makes sense mathematically.

This book covers the basic mathematical ideas, constructions, and methods that come from different fields of mathematical activities and areas of applications that have often evolved in parallel directions. The scope of research is frighteningly broad because it touches on areas that include classical geometry, modern partial differential equations, geometric measure theory, topological groups, and constrained optimization, with applications to classical mechanics of continuous media such as fluid mechanics, elasticity theory, fracture theory, modern theories of optimal design, optimal location and shape of geometric objects, free and moving boundary problems, and image processing. Innovative modeling or new issues raised in some applications force a new look at the fundamentals of well-established mathematical areas such as geometry, to relax basic notions of volume, perimeter, and curvature or boundary value problems, and to find suitable relaxations of solutions. In that spirit, Henri Lebesgue was probably a pioneer when he relaxed the intuitive notion of volume to the one of measure on an equivalence class of measurable sets in 1907. He was followed in that endeavor in the early 1950s by the celebrated work of E. De Giorgi, who used the relaxed notion of perimeter defined on the class of Caccioppoli sets to solve Plateau’s problem of minimal surfaces.

The material that is pertinent to the study of geometric objects and the entities and functions that are defined on them would necessitate an encyclopedic investment to bring together the basic theories and their fields of applications. This objective is obviously beyond the scope of a single book and two authors. The
coverage of this book is more modest. Yet, it contains most of the important funda-
mentals at this stage of evolution of this expanding field.

Even if shape analysis and optimization have undergone considerable and im-
portant developments on the theoretical and numerical fronts, there are still cultural 
barriers between areas of applications and between theories. The whole field is ex-
tremely active, and the best is yet to come with fundamental structures and tools 
beginning to emerge. It is hoped that this book will help to build new bridges and 
stimulate cross-fertilization of ideas and methods.

2 Overview of the Second Edition

The second edition is almost a new book. All chapters from the first edition have 
been updated and, in most cases, considerably enriched with new material. Many 
chapters or parts of chapters have been completely rewritten following the devel-
opments in the field over the past 10 years. The book went from 9 to 10 chapters
with a more elaborate sectioning of each chapter in order to produce a much more 
detailed table of contents. This makes it easier to find specific material.

A series of illustrative generic examples has been added right at the begin-
ing of the introductory Chapter 1 to motivate the reader and illustrate the basic

dilemma: parametrize geometries by functions or functions by geometries? This is 
followed by the big picture: a section on background and perspectives and a more 
detailed presentation of the second edition.

The former Chapter 2 has been split into Chapter 2 on the classical descrip-
tions and properties of domains and sets and a new Chapter 3, where the important

material on Courant metrics and the generic constructions of A. M. Micheletti have
been reorganized and expanded. Basic definitions and material have been added
and regrouped at the beginning of Chapter 2: Abelian group structure on subsets
of a set, connected and path-connected spaces, function spaces, tangent and dual
cones, and geodesic distance. The coverage of domains that verify some segment
property and have a local epigraph representation has been considerably expanded,
and Lipschitzian (graph) domains are now dealt with as a special case.

The new Chapter 3 on domains and submanifolds that are the image of a
fixed set considerably expands the material of the first edition by bringing up the
general assumptions behind the generic constructions of A. M. Micheletti that lead
to the Courant metrics on the quotient space of families of transformations by
subgroups of isometries such as identities, rotations, translations, or flips. The
general results apply to a broad range of groups of transformations of the Euclidean
space and to arbitrary closed subgroups. New complete metrics on the whole spaces
of homeomorphisms and $C^k$-diffeomorphisms are also introduced to extend classical
results for transformations of compact manifolds to general unbounded closed sets
and open sets that are crack-free. This material is central in classical mechanics
and physics and in modern applications such as imaging and detection.

The former Chapter 7 on transformations versus flows of velocities has been
moved right after the Courant metrics as Chapter 4 and considerably expanded. It
now specializes the results of Chapter 3 to spaces of transformations that are
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generated by the flow of a velocity field over a generic time interval. One important motivation is to introduce a notion of semiderivatives as well as a tractable criterion for continuity with respect to Courant metrics. Another motivation for the velocity point of view is the general framework of R. Azencott and A. Trouvé starting in 1994 with applications in imaging. They construct complete metrics in relation with geodesic paths in spaces of diffeomorphisms generated by a velocity field.

The former Chapter 3 on the relaxation to measurable sets and Chapters 4 and 5 on distance and oriented distance functions have become Chapters 5, 6, and 7. Those chapters have been renamed Metrics Generated by \ldots in order to emphasize one of the main thrusts of the book: the construction of complete metrics on shapes and geometries.\footnote{This is in line with current trends in the literature such as in the work of the 2009 Abel Prize winner M. Gromov \cite{Gromov2009} and its applications in imaging by G. Sapiro \cite{Sapiro2009} and F. Mémoli and G. Sapiro \cite{Mémoli2009} to identify objects up to an isometry.} Those chapters emphasize the function analytic description of sets and domains: construction of metric topologies and characterization of compact families of sets or submanifolds in the Euclidean space. In that context, we are now dealing with equivalence classes of sets that may or may not have an invariant open or closed representative in the class. For instance, they include Lebesgue measurable sets and Federer’s sets of positive reach. Many of the classical properties of sets can be recovered from the smoothness or function analytic properties of those functions.

The former Chapter 6 on optimization of shape functions has been completely rewritten and expanded as Chapter 8 on shape continuity and optimization. With meaningful metric topologies, we can now speak of continuity of a geometric objective functional such as the volume, the perimeter, the mean curvature, etc., compact families of sets, and existence of optimal geometries. The chapter concentrates on continuity issues related to shape optimization problems under state equation constraints. A special family of state constrained problems are the ones for which the objective function is defined as an infimum over a family of functions over a fixed domain or set such as the eigenvalue problems. We first characterize the continuity of the transmission problem and the upper semicontinuity of the first eigenvalue of the generalized Laplacian with respect to the domain. We then study the continuity of the solution of the homogeneous Dirichlet and Neumann boundary value problems with respect to their underlying domain of definition since they require different constructions and topologies that are generic of the two types of boundary conditions even for more complex nonlinear partial differential equations. An introduction is also given to the concepts and results from capacity theory from which very general families of sets stable with respect to boundary conditions can be constructed. Note that some material has been moved from one chapter to another. For instance, section 7 on the continuity of the Dirichlet boundary problem in the former Chapter 3 has been merged with the content of the former Chapter 4 in the new Chapter 8.

The former Chapters 8 and 9 have become Chapters 9 and 10. They are devoted to a modern version of the shape calculus, an introduction to the tangential differential calculus, and the shape derivatives under a state equation constraint. In Chapters 3, 5, 6, and 7, we have constructed complete metric spaces of geometries. Those spaces are nonlinear and nonconvex. However, several of them have a group
structure and, in some cases, it is possible to construct $C^1$-paths in the group from velocity fields. This leads to the notion of *Eulerian semiderivative* that is somehow the analogue of a derivative on a smooth manifold. In fact, two types of semiderivatives are of interest: the weaker *Gateaux style semiderivative* and the stronger *Hadamard style semiderivative*. In the latter case, the classical chain rule is still available even for nondifferentiable functions. In order to prepare the ground for shape derivatives, an enriched self-contained review of the pertinent material on semiderivatives and derivatives in topological vector spaces is provided.

The important Chapter 10 concentrates on two generic examples often encountered in shape optimization. The first one is associated with the so-called *compliance problems*, where the shape functional is itself the minimum of a domain-dependent energy functional. The special feature of such functionals is that the adjoint state coincides with the state. This obviously leads to considerable simplifications in the analysis. In that case, it will be shown that theorems on the differentiability of the minimum of a functional with respect to a real parameter readily give explicit expressions of the Eulerian semiderivative even when the minimizer is not unique.

The second one will deal with shape functionals that can be expressed as the saddle point of some appropriate Lagrangian. As in the first example, theorems on the differentiability of the saddle point of a functional with respect to a real parameter readily give explicit expressions of the Eulerian semiderivative even when the solution of the saddle point equations is not unique. Avoiding the differentiation of the state equation with respect to the domain is particularly advantageous in shape problems.

### 3 Intended Audience

The targeted audience is applied mathematicians and advanced engineers and scientists, but the book is also suitable for a broader audience of mathematicians as a relatively well-structured initiation to shape analysis and calculus techniques. Some of the chapters are fairly self-contained and of independent interest. They can be used as lecture notes for a mini-course. The material at the beginning of each chapter is accessible to a broad audience, while the latter sections may sometimes require more mathematical maturity. Thus the book can be used as a graduate text as well as a reference book. It complements existing books that emphasize specific mechanical or engineering applications or numerical methods. It can be considered a *companion* to the book of J. Sokolowski and J.-P. Zolésio [9], *Introduction to Shape Optimization*, published in 1992.

Earlier versions of parts of this book have been used as lecture notes in graduate courses at the Université de Montréal in 1986–1987, 1993–1994, 1995–1996, and 1997–1998 and at international meetings, workshops, or schools: Séminaire de Mathématiques Supérieures on *Shape Optimization and Free Boundaries* (Montréal, Canada, June 25 to July 13, 1990), short course on *Shape Sensitivity Analysis* (Kénitra, Morocco, December 1993), course of the COMETT MATARI European Program on *Shape Optimization and Mutational Equations* (Sophia-Antipolis, France, September 27 to October 1, 1993), CRM Summer School on *Boundaries,
Interfaces and Transitions (Banff, Canada, August 6–18, 1995), and CIME course on Optimal Design (Troia, Portugal, June 1998).

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