# Preface

In this book, we introduce current developments and applications in using iterative methods for solving Toeplitz systems. Toeplitz systems arise in a variety of applications in mathematics, scientific computing and engineering, for instance, numerical partial and ordinary differential equations; numerical solution of convolution-type integral equations; stationary auto-regressive time series in statistics; minimal realization problems in control theory; system identification problems in signal processing and image restoration problems in image processing, see [24, 36, 45, 55, 66].

In 1986, Strang [74] and Olkin [67] proposed independently the use of the preconditioned conjugate gradient (PCG) method with circulant matrices as preconditioners to solve Toeplitz systems. One of the main results of this iterative solver is that the complexity of solving a large class of $n$-by-$n$ Toeplitz systems $T_n\mathbf{u} = \mathbf{b}$ is only $O(n \log n)$ operations. Since then, iterative Toeplitz solvers have garnered much attention and evolved rapidly over the last two decades.

This book is intended to be a short and quick guide to the development of iterative Toeplitz solvers based on the PCG method. Within limited space and time, we are forced to deal with only important aspects of iterative Toeplitz solvers and give special attention to the construction of efficient circulant preconditioners. Applications of iterative Toeplitz solvers to some practical problems will be briefly discussed. We wish that after reading the book, the readers can use our methods and algorithms to solve their own problems easily.

The book is organized into five chapters. In Chapter 1, we first introduce Toeplitz systems and discuss their applications. We give a brief survey of classical (direct) Toeplitz solvers. Some background knowledge of matrix analysis that will be used throughout the book is provided. A preparation for current developments in using the PCG method to solve Toeplitz systems is also given.

In Chapter 2, we study some well-known circulant preconditioners which have been proved to be efficient for solving some well-conditioned Hermitian Toeplitz systems. We introduce the construction of Strang's preconditioner, T. Chan's preconditioner and the superoptimal preconditioner. A detailed analysis of the convergence rate of the PCG method and some numerical tests with these three preconditioners are also given. Other useful preconditioners will be briefly introduced.

Chapter 3 develops a unified treatment of different circulant preconditioners. We consider circulant preconditioners for Hermitian Toeplitz systems from the view point of function theory. Some well-known circulant preconditioners can be derived from convoluting the generating function of the Toeplitz matrix with some famous kernels. Several new circulant preconditioners are then constructed using this approach. An analysis of convergence rate is given with some numerical examples.

Chapter 4 describes how a family of efficient circulant preconditioners can be constructed for ill-conditioned Hermitian Toeplitz systems $T_n\mathbf{u} = \mathbf{b}$. Inspired by the unified theory developed in Chapter 3, the preconditioners are constructed by convoluting the generating function of $T_n$ with the generalized Jackson kernels. When the generating function is nonnegative continuous with a zero of order $2p$, the condition number of $T_n$ is known to grow as $O(n^{2p})$. We show however that the preconditioner is positive definite and the spectrum of the preconditioned matrix is uniformly bounded except for at most $2p + 1$ outliers. Moreover the smallest eigenvalue is uniformly bounded away from zero. Hence the PCG method converges linearly when used to solve the system. Numerical examples are included to illustrate the effectiveness of the preconditioners.

Chapter 5 is devoted to the study of block circulant preconditioners for the solution of block systems $T_{mn}\mathbf{u} = \mathbf{b}$ by the PCG method where $T_{mn}$ are $m$-by-$m$ block Toeplitz matrices with $n$-by-$n$ Toeplitz blocks. Such kind of system appears in many applications especially in image processing [45, 66]. The preconditioners $c_F^{(1)}(T_{mn})$ and $c_{F,F}^{(2)}(T_{mn})$ are constructed to preserve the block structure of $T_{mn}$ and are defined to be the minimizers of $\|T_{mn} - C_{mn}\|_{\mathscr{F}}$ over some special classes of matrices. We prove that if $T_{mn}$ is positive definite, then $c_F^{(1)}(T_{mn})$ and $c_{F,F}^{(2)}(T_{mn})$ are positive definite too. We illustrate their effectiveness for solving block Toeplitz systems by some numerical examples.

To facilitate the use of the methods discussed in this book, we have included in the Appendix the Matlab programs that were used to generate our numerical results here.

This book contains some important parts of our research work in the past 20 years. Some research results are joint work with Prof. Michael K. Ng of the Department of Mathematics, Hong Kong Baptist University; Prof. Man-Chung Yeung of the Department of Mathematics, University of Wyoming; and Dr. Andy M. Yip of the Department of Mathematics, National University of Singapore. We are indebted to Prof. Tony F. Chan of the Department of Mathematics, University of California at Los Angeles; Prof. Gene Golub of the Department of Computer Science, Stanford University; Prof. Robert Plemmons of the Department of Mathematics and Computer Science, Wake Forest University; and Prof. Gilbert Strang of the Department of Mathematics, Massachusetts Institute of Technology, for their enlightening ideas, suggestions and comments, from which we benefited a great deal. We are grateful to Prof. Furong Lin of Department of Mathematics, Shantou University; and our student Mr. Wei Wang, for their help in the Matlab programs used in the book. Thanks are also due to our families for their encouragement, great support and patience which are essential to the completion of the book.

The research results included in the book are supported by a number of grants provided by HKRGC of Hong Kong and the research grant 050/2005/A provided by FDCT of Macao.