Origin of the Finite Element Method

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→ (Rayleigh –) Ritz – Galerkin Method
Literature

- google: > 10000000 pages
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- www.web-spline.de (K. Höllig, U. Reif, J. Wipper)
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related method, using b-splines:
J.A. Cottrell, T.J.R. Hughes, Y. Bazilevs, Isogeometric Analysis, John Wiley & Sons Ltd., 2009.
History of Finite Elements and Splines

Engineering

Turner, Clough, Martin, and Topp (1956)
Argyris (1960)
Clough (1960)

de Casteljau (1959)
Bezier (1966)

FEM

Rayleigh (1870)
Ritz (1908)
Galerkin (1915)
Courant (1943)
Strang and Fix (1973)

Splines

Schoenberg (1946)
de Boor (1972)

Mathematics
Splines as Finite Elements

grid with inner and outer B-splines
principal difficulties

- essential boundary conditions

\[ \sum_{k} u_k b_k = 0 \text{ on } \partial D \implies u_k = 0, \ k \sim \partial D \]

\[ \implies \text{ poor approximation order} \]
principal difficulties

- essential boundary conditions

\[ \sum_{k} u_k b_k = 0 \text{ on } \partial D \implies u_k = 0, \ k \sim \partial D \]

\[ \leadsto \text{poor approximation order} \]

- stability

\[ \| c_k \| \not\leq \| \sum_{k} c_k b_k \| \quad (h \to 0) \]

\[ \leadsto \text{ill-conditioned systems, slow convergence of iterative schemes} \]
Weighted Extended B-Splines

homogeneous boundary conditions, modeled with a weight function

\[ b_k \rightarrow wb_k, \quad k \in K \]

suggested by Kantorovich and Krylow, studied by Rvachev
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stabilization via extension of inner B-splines

\[ b_i \rightarrow b_i + \sum_{j \in J(i)} e_{i,j} b_j, \quad i \in I \]

based on Marsden’s identity
Weighted Extended B-Splines

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based on Marsden’s identity

\[ \leadsto \text{weighted extended B-splines (web-splines)} \]

\[ B_i = \gamma_i w \left( b_i + \sum_{j \in J(i)} e_{i,j} b_j \right) \]

with standard properties of finite elements
Advantages of WEB-Splines

flexibility of mesh-based elements and computational efficiency of B-splines
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- meshless method
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Introduction
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- compatibility with CAD/CAM systems
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- skipping dependencies on parameters

\[ b_k = b_{k,h}^n, \ldots \]
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  \[ \leq, \geq, \preccurlyeq \]

- spline approximation with coefficient vector \[ U = \{ u_k \}_{k \in K} \]
  \[ u \approx u_h = \sum_k u_k b_k, \]
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  \[ \preceq, \succeq, \simeq \]

- spline approximation with coefficient vector \( U = \{u_k\}_{k \in K} \)
  \[ u \approx u_h = \sum_k u_k b_k, \]

- vectors and matrices
  \[ G = \{g_{k,i}\}_{k,i \in I} \]
  products \( UV \) without transposition