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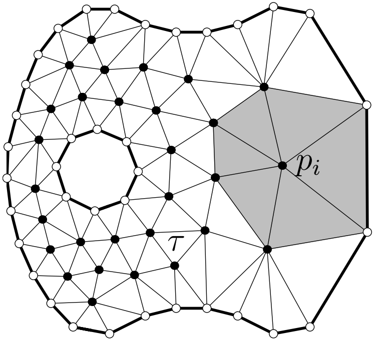

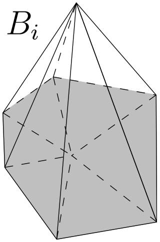

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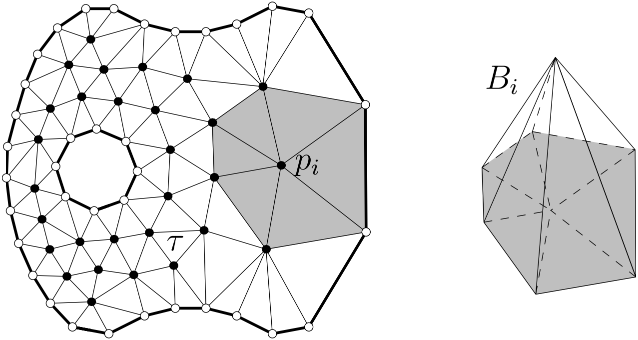

approximation, determined by Lagrange data

$$u_h = \sum_{i \in I} u_i B_i, \quad u_i = u_h(p_i)$$

Ritz-Galerkin System for Hat–Functions

system entries: sum contributions from each triangle $\tau = [p_i, p_j, p_k]$

$$g_{\ell,m} = \sum_{\tau} \int_{\tau} \operatorname{grad} B_\ell \, \operatorname{grad} B_m, \quad f_\ell = \sum_{\tau} \int_{\tau} f \, B_\ell$$

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compute gradients via directional derivatives

$$\underbrace{\begin{pmatrix} \operatorname{grad} B_i \\ \operatorname{grad} B_j \\ \operatorname{grad} B_k \end{pmatrix}}_{G_\tau} \begin{pmatrix} p_j - p_i & p_k - p_j \end{pmatrix} = R, \quad R = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$$

G: add submatrix of

$$\text{area}(\tau)\, G_\tau G_\tau^t = \frac{|\det P|}{2}\, RP^{-1}(P^t)^{-1}R^t\,,$$

corresponding to inner vertices

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F: add subvector of

$$\frac{|\det P|}{6} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} f_i \\ f_j \\ f_k \end{pmatrix}$$

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Details

Details

exact Taylor approximation for linear functions

$$B(q) = B(p) + \underbrace{\operatorname{grad} B\,(q-p)}_{\text{directional derivative}}$$

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quadrature formula for triangle $\tau = [p_i, p_j, p_k]$

$$\int_\tau g = \frac{\operatorname{area}\tau}{3}\,\left[g((p_i+p_j)/2) + g((p_j+p_k)/2) + g((p_k+p_i)/2)\right]$$

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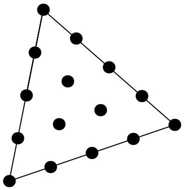
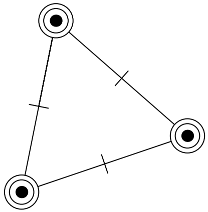
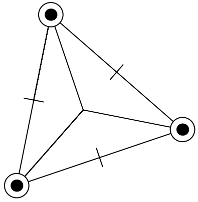
exact for quadratic polynomials $\leadsto$ error $O(h^5)$, $h = \operatorname{diam}\tau$
apply with $g = fB_\ell$ and linear interpolation of f and $B_\ell \leadsto$

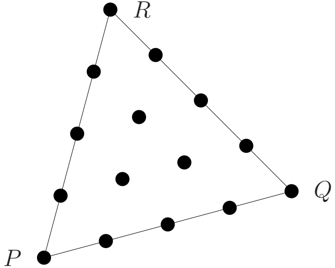
$$[\ldots] = f_\ell/2 + f_m/4 + f_{m'}/4, \quad m, m' \neq \ell\,,$$

since $g((p_\ell + p_m)/2) = ((f_\ell + f_m)/2)\,((1+0)/2)$

Bivariate Finite Elements

		
Lagrange	Argyris	Clough–Tocher
degree 4, $\in C^0$ dimension 15	degree 5, $\in C^1$ dimension 21	degree 3, $\in C^1$ dimension 12

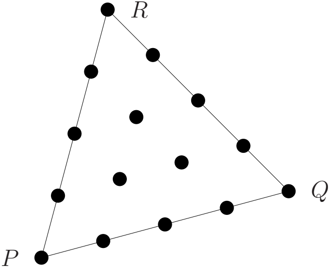
Lagrange Elements



nodes, labeled (i, j, k) with $i + j + k = n$

$$\frac{i}{n}P + \frac{j}{n}Q + \frac{k}{n}R$$

Lagrange Elements



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$$B_{i,j,k} = \binom{n}{i} u^0 \cdots u^{i-1} \binom{n}{j} v^0 \cdots v^{j-1} \binom{n}{k} w^0 \cdots w^{k-1}$$

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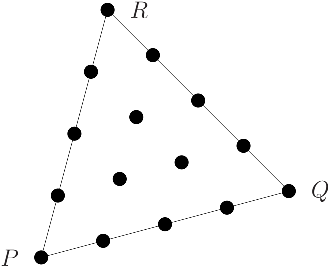

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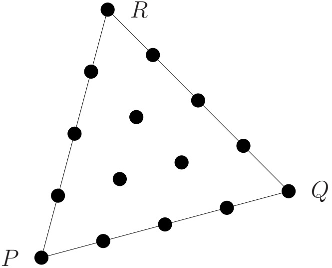

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$\rightsquigarrow$ interpolation of Lagrange data

Details

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(view i as variable, ranging from ℓ to n)

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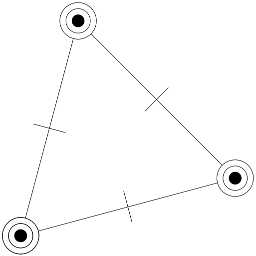
(view i as variable, ranging from ℓ to n)
value of $u^0 u^1 \cdots u^{i-1}$ at X

$$\frac{i}{n} \frac{i-1}{n-1} \cdots \frac{1}{n-i+1} = \binom{n}{i}^{-1}$$

$\implies B_{i,j,k}(X) = 1$

Argyris Triangle



degree 5, $\in C^1$
dimension 21

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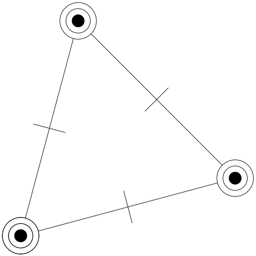

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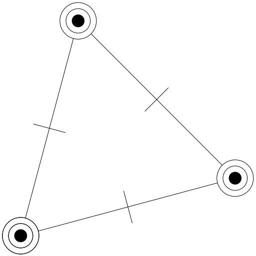

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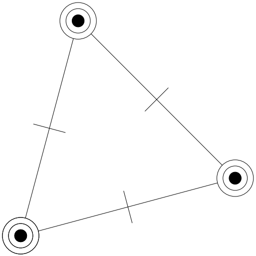

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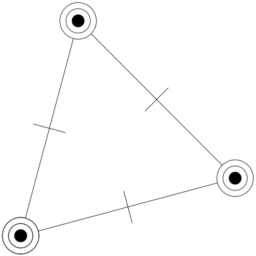

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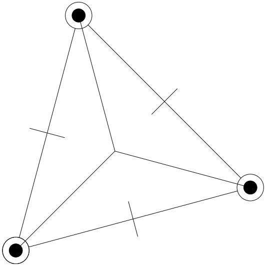
$\Longrightarrow$ values and derivatives prescribed at triangle boundaries
(quintic and quartic polynomials, respectively)

Clough-Tocher Macro-Triangle



degree 3, $\in C^1$
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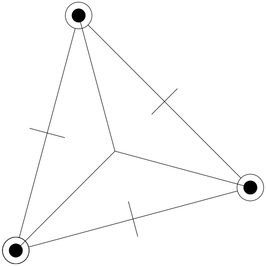
Clough-Tocher Macro-Triangle



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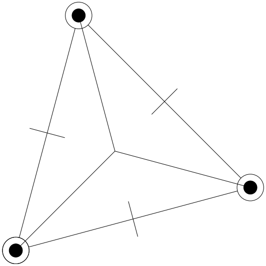

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defining data

- values and gradients at vertices

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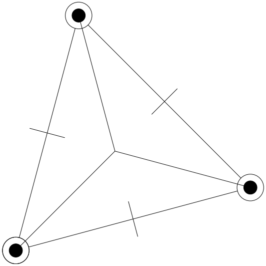

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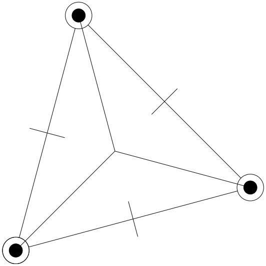

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defining data

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- normal derivatives at edge mid-points

$\Longrightarrow$ values and derivatives prescribed along the outer boundaries (cubic and quadratic polynomials, respectively)

Properties of Finite Elements

The basis functions B_i of standard mesh-based finite element subspaces are piecewise polynomials of degree $\leq n$ with support on few neighboring mesh cells. They are at least continuous and compatible with homogeneous boundary conditions on piecewise linear boundaries.

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No mesh generation is required, accurate smooth approximations are possible with relatively low-dimensional finite element subspaces, and boundary conditions are satisfied exactly.