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$$f = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} f_k \varphi_k, \quad \varphi_k(x) = \sin(\pi k_1 x_1) \sin(\pi k_2 x_2)$$

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comparision of coefficients $\rightsquigarrow$

$$u = \sum_{k_1} \sum_{k_2} u_k \varphi_k, \quad u_k = \frac{1}{\pi^2(k_1^2 + k_2^2)} f_k$$

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convergence in the scalar product norm

$$f \in L_2(D) \Leftrightarrow \|f\|_0^2 = \int_D |f|^2 = \frac{1}{4} \sum_{k_1} \sum_{k_2} |f_k|^2 < \infty$$

sine coefficients of second-order partial derivatives $\partial_\nu \partial_\mu u$:

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Poisson energy functional

$$\begin{aligned}\mathcal{Q}(u) &= \frac{1}{2} \int_D (|\partial_1 u|^2 + |\partial_2 u|^2) - \int_D fu \\ &= \frac{\pi^2}{8} \sum_{k_1} \sum_{k_2} (k_1^2 + k_2^2)|u_k|^2 - \frac{1}{4} \sum_{k_1} \sum_{k_2} f_k u_k\end{aligned}$$

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\end{aligned}$$

well defined for functions with square integrable first-order partial derivatives:

$$\partial_\nu u \in L_2(D), \quad \nu = 1, 2$$

Sobolev Spaces

The Sobolev space $H^\ell(D)$ consists of all functions u for which the partial derivatives of order $\leq \ell$,

$$\partial^\alpha u = \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m} u, \quad |\alpha| = \alpha_1 + \cdots + \alpha_m \leq \ell\,,$$

are square integrable. It is a Hilbert space with the scalar product

$$\langle u, v \rangle_\ell = \sum_{|\alpha| \leq \ell} \int_D \partial^\alpha u \, \partial^\alpha v\,.$$

In addition to the induced norm $\|u\|_\ell = \sqrt{\langle u, u \rangle_\ell}$ the standard semi-norm on H^ℓ is defined as

$$|u|_\ell = \left(\sum_{|\alpha| = \ell} \int_D |\partial^\alpha u|^2 \right)^{1/2},$$

i.e., it involves only derivatives of the highest order.

Weak Derivatives

An integrable function $\partial^\alpha u$ is a weak derivative of u on a domain D if

$$\int_D (\partial^\alpha u)\varphi = (-1)^{|\alpha|} \int_D u(\partial^\alpha \varphi)$$

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The definition is consistent with classical definition for smooth functions u; it just restates the formula for integration by parts in this case.

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radial function on the unit ball

$$u(x) = r^p, \quad r = \sqrt{x_1^2 + \cdots + x_m^2} < 1,\ p \neq 0$$

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integrable for $p > 1 - m$ since

$$|\partial_\nu u| \leq c r^{p-1}$$

weak derivative despite singularity since, for φ with compact support in D,

$$\begin{aligned}\int_D \partial_\nu u\,\varphi &= \int_{B_\varepsilon} \ldots + \int_{D\setminus B_\varepsilon} \ldots \\ &= \left[\int_{B_\varepsilon} \partial_\nu u\,\varphi - \int_{\partial B_\varepsilon} u\,\varphi\,\xi_\nu\right] - \int_{D\setminus B_\varepsilon} u\partial_\nu\varphi\end{aligned}$$

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$$\begin{aligned}\|u\|_1^2 &= \int_D |u|^2 + \sum_\nu \int_D |\partial_\nu u|^2 \\ &= c\int_0^1 \left(r^{2p} + p^2 r^{2p-2}\right) r^{m-1}\,dr\end{aligned}$$

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negative exponent p possible for $m > 2$
$\rightsquigarrow$ unbounded functions with weak derivatives

Example

piecewise constant function on $D = (-1, 1)^m$:

$$u(x) = \begin{cases} 1 & \text{for } x_1 \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

discontinuous across the hyperplane $S = \{x \in \mathbb{R}^m : \ x_1 = 0\}$

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choose

$$\varphi(x) = \psi(x_1/\varepsilon, x_2, \ldots, x_m) \geq 0 \quad (\varepsilon > 0)$$

with $\psi(0) > 0$

$\rightsquigarrow$ right side equals

$$-\int_{x_1 \geq 0} \partial_1 \varphi = \int_{(-1,1)^{m-1}} \psi(0, x_2, \ldots, x_m)\, dx_2 \cdots dx_m = \text{const} > 0$$

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left side tends to zero:

$$\left| \int_D v\varphi \right| \leq \|v\|_0 \left(\int_{\mathbb{R}^m} |\psi(x_1/\varepsilon, x_2, \ldots, x_m)|^2 \, dx \right)^{1/2} = O(\varepsilon^{1/2})$$

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$\rightsquigarrow$ so-called generalized derivatives

Sobolev Spaces with Boundary Conditions

The subspace $H_0^\ell(D) \subset H^\ell(D)$ consists of all functions which vanish on ∂D. More precisely, $H_0^\ell(D)$ is the closure of all smooth functions with compact support in D with respect to the norm $\|\cdot\|_\ell$.

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Boundary values are defined by a limit process. The trace operator

$$u \mapsto u_{|\partial D}$$

has a continuous extension from smooth functions to $H^\ell(D)$.