Orthogonality Relation

The Ritz-Galerkin approximation $u_h \in \mathbb{B}_h \subset H$ of a solution $u \in H$ to the variational equations

$$\forall v \in H : \; a(u, v) = \lambda(v)$$

is defined by

$$\forall v_h \in \mathbb{B}_h : \; a(u_h, v_h) = \lambda(v_h) \, .$$

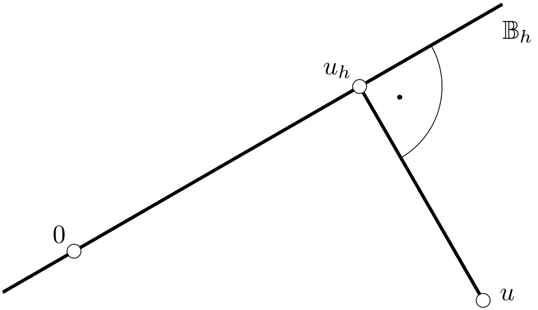
As a consequence (subtracting the two equations with $v = v_h = w_h$) the error satisfies the orthogonality relation

$$a(u - u_h, w_h) = 0 \quad \forall w_h \in \mathbb{B}_h \, .$$

symmetric bilinear form a:

orthogonality $\Leftrightarrow$ best approximation with respect to scalar product norm symmetric bilinear form *a*:

orthogonality ⇔ best approximation with respect to scalar product norm



$u - u_h \perp_a \mathbb{B}_h$

Céa's Inequality

The error of the Ritz–Galerkin approximation $u_h \approx u$ for an elliptic bilinear form a satisfies

$$\|u - u_h\| \leq (c_b/c_e) \inf_{v_h \in \mathbb{B}_h} \|u - v_h\|,$$

where c_b and c_e are the ellipticity constants.

Proof

orthogonality relation

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$w_h = u_h - v_h \rightsquigarrow$

$$a(u - u_h, u - u_h) = a(u - u_h, u - v_h), \quad v_h \in \mathbb{B}_h$$

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ellipticity $\implies$

$$c_e \|u - u_h\|^2 \leq \text{left side}, \quad \text{right side} \leq c_b \|u - u_h\| \|u - v_h\|$$

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cancel common factor $\|u - u_h\| \rightsquigarrow$ estimate for $\|u - u_h\|$

Example

piecewise linear Ritz–Galerkin approximation of

$$-\Delta u = f \text{ in } D, \quad u = 0 \text{ on } \partial D$$

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$$\|u - u_h\|_1 \leq (c_b/c_e) \inf_{v_h} \|u - v_h\|_1$$

for a quasi-uniform boundary conforming triangulation of a convex domain

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$$\inf_{v_h} \|u - v_h\|_1 \leq c_a\, h \|u\|_2$$

with h the mesh width of the triangulation
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$$\|u - u_h\|_1 \le c_1 h \|f\|_0$$

with $c_1 = (c_b/c_e) c_a c_r$

Aubin–Nitsche Duality Principle

If H is a subspace of a Hilbert space H_*, the error $e_h = u - u_h$ of the Ritz–Galerkin approximation satisfies

$$\|e_h\|_*^2 \leq c_b\, r\, \|e_h\|, \quad r = \inf_{v_h \in \mathbb{B}_h} \|u_* - v_h\|,$$

where u_* is the solution of the dual problem

$$a(v, u_*) = \langle v, e_h \rangle_*, \quad v \in H,$$

and $\langle \cdot, \cdot \rangle_*$ denotes the scalar product on H_*.

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boundedness of $a \implies$

$$\text{right side} \le c_b \, \|e_h\| \, \|u_* - v_h\|$$

for any $v_h \in \mathbb{B}_h$

Example

Poisson's problem on a convex polygonal domain

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u_h: approximation with piecewise linear finite elements on a quasi-uniform boundary conforming triangulation

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u_h: approximation with piecewise linear finite elements on a quasi-uniform boundary conforming triangulation
Aubin–Nitsche duality principle $\Longrightarrow$

$$\|e_h\|_0^2 \le c_b \left[\inf_{v_h} \|u_* - v - h\|_1 \right] \|e_h\|_1 \,,$$

with $H = H_0^1(D)$, $H_* = L_2(D)$, $\|\cdot\|_* = \|\cdot\|_0$,
and u_* solution of the dual problem

$$a(v, u_*) = \int_D \operatorname{grad} v \operatorname{grad} u_* = \int_D v e_h$$

error estimate $\rightsquigarrow$

$$[\ldots] \leq c_a h \|u_*\|_2$$

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$$\| u_* \|_2 \le c_r \| e_h \|_0$$

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$$\|e_h\|_0^2 \leq c_b [c_a h c_r \|e_h\|_0) c_1 h \|f\|_0$$

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$\Longrightarrow$

$$\|e_h\|_0 \leq c_0 h^2 \|f\|_0 \,, \; c_0 = c_b c_a c_r c_1$$

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optimal order for spline-based elements of degree n

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best approximation

$$\inf_{v_h} \|u - v_h\|_\ell \leq c_s h^{n+1-\ell} \|u\|_{n+1}$$

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$\|u_*\|_2 \leq c_r \|e_h\|_0 \rightsquigarrow \|e_h\|_0 \preceq h^{n+1}$