

Contents

List of Figures	xv
List of Tables	xvii
List of Algorithms	xix
Preface	xxi
I Elliptic Problems	1
1 One-dimensional problem	3
1.1 Model problem	3
1.2 A class of DG methods	3
1.3 Existence and uniqueness of the DG solution	6
1.4 Linear system	7
1.4.1 Computing the matrix A	8
1.4.2 Computing the right-hand side \mathbf{b}	10
1.4.3 Imposing boundary conditions strongly	11
1.5 Convergence of the DG method	12
1.6 Numerical experiments	13
1.7 Bibliographical remarks	15
Exercises	17
2 Higher dimensional problem	19
2.1 Preliminaries	19
2.1.1 Vector notation	19
2.1.2 Sobolev spaces	19
2.1.3 Trace theorems	22
2.1.4 Approximation properties	24
2.1.5 Green's theorem	24
2.1.6 Cauchy–Schwarz's and Young's inequalities	25
2.2 Model problem	25
2.2.1 Weak solution	26
2.2.2 Numerical solution	26

2.3	Broken Sobolev spaces	27
2.3.1	Jumps and averages	28
2.4	Variational formulation	29
2.4.1	Consistency	30
2.5	Finite element spaces	32
2.5.1	Reference elements versus physical elements	32
2.5.2	Basis functions	35
2.5.3	Numerical quadrature	36
2.6	DG scheme	37
2.7	Properties	38
2.7.1	Coercivity of bilinear forms	38
2.7.2	Continuity of bilinear form	40
2.7.3	Local mass conservation	41
2.7.4	Existence and uniqueness of DG solution	42
2.8	Error analysis	42
2.8.1	Error estimates in the energy norm	42
2.8.2	Error estimates in the L^2 norm	46
2.9	Implementing the DG method	49
2.9.1	Data structure	49
2.9.2	Local matrices and right-hand sides	51
2.9.3	Global matrix and right-hand side	55
2.10	Numerical experiments	57
2.10.1	Smooth solution	57
2.10.2	Singular solution	58
2.10.3	Condition number	59
2.11	The local discontinuous Galerkin method	59
2.11.1	Definition of the mixed DG method	60
2.11.2	Existence and uniqueness of the solution	62
2.11.3	A priori error estimates	63
2.12	DG versus classical finite element method	64
2.13	Bibliographical remarks	67
	Exercises	67
 II Parabolic Problems		69
 3 Purely parabolic problems		71
3.1	Preliminaries	71
3.1.1	Functional spaces	71
3.1.2	Gronwall's inequalities	71
3.1.3	Taylor's expansions	72
3.1.4	Poincaré's inequalities	72
3.1.5	Inverse inequalities	73
3.2	Model problem	73
3.3	Semidiscrete formulation	73
3.3.1	A priori bounds	75

Contents	xi
3.3.2	Error estimates 77
3.4	Fully discrete formulation 80
3.4.1	Backward Euler discretization 80
3.4.2	Forward Euler discretization 84
3.4.3	Crank–Nicolson discretization 86
3.4.4	Runge–Kutta discretization 87
3.4.5	DG in time discretization 88
3.5	Implementation 91
3.6	Bibliographical remarks 92
Exercises 92
4	Parabolic problems with convection 95
4.1	Model problem 95
4.2	Semidiscrete formulation 96
4.2.1	Existence and uniqueness of solution 97
4.2.2	Consistency 97
4.2.3	Error estimates 98
4.3	Fully discrete formulation 100
4.3.1	Overshoot and undershoot 100
4.3.2	Slope limiters 101
4.3.3	An improved DG method 104
4.4	Bibliographical remarks 106
Exercises 106
III	Applications 107
5	Linear elasticity 109
5.1	Preliminaries 109
5.1.1	Strain and stress tensors 109
5.1.2	Korn’s inequalities 110
5.2	Model problem 110
5.3	DG scheme 111
5.3.1	Consistency 112
5.3.2	Local equilibrium 112
5.3.3	Coercivity 112
5.4	Error analysis 113
5.5	Bibliographical remarks 115
Exercises 115
6	Stokes flow 117
6.1	Preliminaries 117
6.1.1	Vector notation 117
6.1.2	Barycentric coordinates 117
6.1.3	An approximation operator of degree one 119
6.1.4	An approximation operator of higher degree 120

6.1.5	Local L^2 projection	121
6.1.6	General inf-sup condition	121
6.2	Model problem and weak solution	122
6.3	DG scheme	123
6.3.1	Existence and uniqueness of solution	124
6.3.2	Local mass conservation	124
6.4	Discrete inf-sup condition	125
6.5	Error estimates	126
6.6	Numerical results	129
6.7	Bibliographical remarks	129
	Exercises	130
7	Navier–Stokes flow	131
7.1	Preliminaries	131
7.1.1	Sobolev imbedding	131
7.1.2	Hölder’s inequality	132
7.1.3	Brouwer’s fixed point theorem	132
7.2	Model problem and weak solution	133
7.3	DG discretization	133
7.3.1	Nonlinear convective term	134
7.3.2	Scheme	136
7.3.3	Consistency	136
7.4	Existence and uniqueness of solution	136
7.4.1	Existence of discrete velocity	136
7.4.2	Existence of discrete pressure	137
7.4.3	A priori bounds	138
7.4.4	Uniqueness	138
7.5	A priori error estimates	139
7.6	Numerical experiments	140
7.6.1	Effects of penalty size	140
7.6.2	Step channel problem	141
7.7	Bibliographical remarks	143
8	Flow in porous media	145
8.1	Two-phase flow	145
8.1.1	Model problem	146
8.1.2	A sequential approach	148
8.1.3	A coupled approach	150
8.1.4	Numerical examples	151
8.2	Miscible displacement	153
8.2.1	Semidiscrete formulation	155
8.2.2	A fully discrete approach	156
8.2.3	Numerical examples	157
8.3	Bibliographical remarks	158

Contents	xiii	
A	Quadrature rules	159
A.1	Gauss quadrature rule on intervals	159
A.2	Quadrature rules on the reference triangle	159
A.3	Quadrature rule on the reference quadrilateral	161
B	DG codes	163
B.1	A MATLAB implementation for a one-dimensional problem	163
B.2	Selected C routines for higher dimensional problem	165
C	An approximation result	175
	Bibliography	179
	Index	189