Preface

> *I thought that instead of the great number of precepts of which logic is composed, I would have enough with the four following ones, provided that I made a firm and unalterable resolution not to violate them even in a single instance. The first rule was never to accept anything as true unless I recognized it to be certainly and evidently such …. The second was to divide each of the difficulties which I encountered into as many parts as possible, and as might be required for an easier solution.* (Descartes)

On one level, this text can be viewed as suitable for a traditional course on ordinary differential equations. Since differential equations are the basis for models of any physical systems that exhibit smooth change, students in all areas of the mathematical sciences and engineering require the tools to understand the methods for solving these equations. It is traditional for this exposure to start during the second year of training in calculus where the basic methods of solving one and two-dimensional (primarily linear) ODEs are studied. The typical reader of this text will have had such a course, as well as an introduction to analysis where the theoretical foundations (the ε's and δ's) of calculus are elucidated. The material for this text has been developed over a decade in a course given to upper division undergraduates and beginning graduate students in applied mathematics, engineering and physics at the University of Colorado. In a one-semester course, I typically cover most of the material in Chapters 1-6 and add a selection of sections from later chapters.

There are a number of classic texts for a traditional differential equations course, for example (Coddington and Levinson 1955; Hirsch and Smale 1974; Hartman 2002). Such courses usually begin with a study of linear systems; we begin there as well in Chapter 2. Matrix algebra is fundamental to this treatment, so we give a brief discussion of eigenvector methods and an extensive treatment of the matrix exponential. The next stage in the traditional course is to provide a foundation for the study of nonlinear differential equations by showing that, under certain conditions, these equations have solutions (existence) and that there is only one solution that satisfies a given initial condition (uniqueness). The theoretical underpinning of this result, as well as many other results in applied mathematics, is the majestic contraction mapping theorem. Chapter 3 provides a self-contained introduction to the analytic foundations needed to understand this theorem. Once this tool is concretely understood, many proofs quickly yield to its power. It is possible to omit §3.3-3.5 as most of the material is not heavily used in later chapters, though at least passing acquaintance with Thm 3.10 and Lemma 3.13 (Grönwall) is to be encouraged.

However, this text does not aim to cover only the material in such a traditional ODE course; rather, it aspires to serve as an introduction to the more modern theory of dynamical systems. The emphasis is on obtaining a *qualitative* understanding of the properties of *differential* dynamical systems, namely those evolution rules that describe smooth evolution in time.[1] The primary concept of this study, the *flow*, is introduced in Chapter 4. The qualitative theory is often concerned with questions of shape and asymptotic behavior that lead us to use topological notions such as conjugacy in the classification of dynamics.

The classification of dynamical behavior begins with the simplest orbits, equilibria and periodic orbits. As Henri Poincaré noted in his classic *Les Methodes Nouvelles de la Mechanique Celeste*,

> *what renders these periodic solutions so precious to us is that they are, so to speak, the only breach through which we may attempt to penetrate an area hitherto deemed inaccessible* (Poincaré, 1892, Vol 1 §36).

It is only with the demonstration that dynamics in the neighborhood of some of these orbits is conjugate to their linearization that the predisposition of applied scientists to concentrate on linear systems is seen to have any value whatsoever.

The local classification of equilibria leads to the theory of invariant manifolds in Chapter 5. The stable and unstable manifolds, proved to exist for a hyperbolic saddle, give rise to one prominent mechanism for chaos—heteroclinic intersection. The center manifold theorem is also important preparation for the treatment of bifurcations in Chapter 8.

As mathematicians, allow yourselves to become entranced by the exceptions to the validity of linearization, namely with those orbits that are nonhyperbolic. It is in the study of these exceptions that we find the most beautiful dynamics—even in the case of the phase plane, to which we return in Chapter 6. The first three sections of this chapter are fundamental; §6.4-6.8 can be omitted in favor of later chapters. As we see in Chapter 8, the exceptional cases form the organizing centers for the behavior of systems undergoing changing parameters. A qualitative change in the behavior under a small change of parameters is called a bifurcation. A complete exegesis of theory of bifurcations requires a full text on its own; and there are many excellent texts appropriate for a more advanced class (Guckenheimer and Holmes 1983; Golubitsky and Schaeffer 1985; Kuznetsov 1995). We introduce the reader to the basic ideas of normal forms and treat codimension-one and two bifurcations.

Perhaps the most exciting recent developments in dynamical systems are those that show that even simple systems can behave in complicated ways, namely the phenomena of *chaos*. In Chapter 7, we introduce the reader to the necessary concepts to understand chaos: Lyapunov exponents, transitivity, fractals, etc. We also give an extensive discussion of Melnikov's method for the onset of chaos in Chapter 8. A more advanced treatment of chaotic dynamics requires a discussion of discrete dynamics (mappings) and can be found in texts such as (Katok and Hasselblatt 1999; Robinson 1999; Wiggins 2003).

The final chapter treats the subject closest to this author's heart: Hamiltonian dynamics. Since the basic models of physics all have a Hamiltonian (or Lagrangian) formulation, it is worthwhile to become familiar with them. While a traditional physics text treats these on a concrete level, we provide an introduction to some of the geometrical aspects of Hamil-

[1] This is not to say that the dynamical systems that we study are always *differentiable*—vector fields need not be smooth.

tonian dynamics, including a discussion of their variational foundation, spectral properties, the KAM theorem, and transition to chaos. Again, there are several advanced texts that go much farther, for example (Arnold 1978; Lichtenberg and Lieberman 1992; Meyer and Hall 1992).

While the proofs of many of the classical theorems are included, this text is not just an abstract treatment of ODEs, but attempts to place the theory in the context of its many applications to physics, biology, chemistry and engineering. Examples in such areas as population modeling, fluid convection, electronics, and mechanics are discussed throughout the text, and especially in Chapter 1. The exercises introduce the reader to many more. Furthermore, to develop a geometrical understanding of dynamics, each student must experiment; we provide some examples of simple codes written in Maple, Mathematica and Matlab in the Appendix, and use the exercises to encourage the student to explore further. There are several texts that focus completely on using one or more of tools like these to explore dynamics (Lynch 2001; Baumann 2004).

I hope that this book conveys a bit my amazement with the beauty and utility of this field. Dynamical systems is the perfect combination of analysis, geometry and physical intuition. Central questions in dynamics have been formulated for centuries, have been solved in the past few years, and await solution by the next generation.

> *It is far better to foresee even without certainty than not to foresee at all.* (Poincaré)

James Meiss
Boulder, Colorado
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