Preface

Wave phenomena are abundant in nature. Familiar examples include water waves and optical waves. Low-amplitude waves are governed by linear partial differential equations. A main feature of linear wave phenomena is dispersion, i.e., different Fourier modes inside a disturbance travel at different speeds. This often leads to the spreading and decay of a local disturbance. For instance, if we throw a stone into a pond, concentric rings of ripples will spread out. Outer ripples generally have higher wavenumbers (shorter spatial periods) and travel faster. After a short time, all the ripples will disperse and disappear (even in the absence of dissipation). Similarly, when a low-intensity light beam passes through the air or a crystal, the beam broadens over distance. This phenomenon is called diffraction in optics, and it is the counterpart of dispersion in water waves.

In 1834, Scott Russell accidentally observed a type of water wave which did not disperse. Below is his original description:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth, and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation" (Russell (1844)).

This phenomenon was recreated on the Union Canal near Edinburgh in July 1995, and a photo is shown in Fig. 1.

Russell's observation puzzled physicists for a long time and caused much controversy, because it could not be explained by linear water wave theory. In 1895, Diederik

Figure 1. *Recreation of Russell's* 1834 *observation of solitary water waves in a canal. (Image courtesy of Mathematics Department, Heriot-Watt University, Edinburgh, Scotland.)*

Korteweg and Gustav de Vries considered this problem. They noticed that while dispersion causes a water wave to decay, nonlinear effects can cause it to steepen—a phenomenon we always see when waves run up a beach. When self-steepening balances dispersion, a solitary wave which moves without change of shape (i.e., the "wave of translation") can form. Performing detailed theoretical analysis, Korteweg and de Vries (1895) derived the following nondimensionalized wave equation which is now called the Korteweg–de Vries (KdV) equation:

$$\psi_t + \psi_{xxx} + 6\psi\psi_x = 0. \tag{1}$$

Here ψ is the elevation of the water surface. This equation admits traveling solitary waves

$$\psi(x,t) = \frac{1}{2}c\,\mathrm{sech}^2\frac{1}{2}\sqrt{c}\,(x-ct), \tag{2}$$

where c is the wave speed. These solitary wave solutions correspond to the wave of translation in Russell's observation.

Not much progress was made on the KdV equation until the 1960s, when Zabusky and Kruskal (1965) numerically discovered the elastic collision between KdV solitary waves, and then Gardner, et al. (1967) invented the inverse scattering transform method and solved the KdV equation analytically. This pioneering work initiated an unprecedented burst of research activities on nonlinear waves. In subsequent years, many other nonlinear equations such as the nonlinear Schrödinger (NLS) equation, the sine-Gordon equation, and the Kadomtsev–Petviashvili (KP) equation were solved by this method, and such equations are now called integrable equations (see Ablowitz and Segur (1981), Zakharov et al. (1984), Newell (1985), Faddeev and Takhtadjan (1987), Ablowitz and Clarkson (1992) for reviews). In addition, the theory of integrable equations was also greatly expanded along many directions such as the Riemann–Hilbert formulation (Zakharov et al. (1984)) and the direct methods (Hirota (2004)). Interesting developments on integrable equations continue till this day.

While integrable equations constitute an important part of the nonlinear wave theory, most nonlinear wave equations encountered in physics and engineering are not integrable. The study of nonintegrable equations started as early as the 1970s, and intensified from the 1990s. These investigations reveal that solution dynamics in nonintegrable equations can be much richer and more complex. For instance, in nonintegrable equations, solitary waves can be unstable; their interactions can exhibit fractal scattering phenomena; solitary waves can be embedded inside the continuous spectrum and thus exhibit unique dynamical properties; solitary waves can suffer critical collapse in multidimensions, etc. Many exciting theories have also been developed, such as the Vakhitov–Kolokolov criterion for linear stability of solitary waves (Vakhitov and Kolokolov (1973)), the exponential asymptotics method for calculating nonlocal solitary waves (see Boyd (1998) for a review), the virial theorem and the wave collapse theory (see Sulem and Sulem (1999) for a review), and the theories for fractal scatterings in solitary wave interactions (Goodman and Haberman (2007), Zhu et al. (2008a)).

In nonlinear wave studies, numerical computations play an important role. This is particularly so for nonintegrable equations. Numerical computations include evolution simulation of initial value problems, computation of solitary wave solutions, computation of stability spectra of solitary waves, etc. In early computations, finite difference methods were often used. In recent years, a wide range of spectrally accurate numerical methods were developed. Examples include the split-step method for evolution simulations, the Newton-conjugate-gradient method for solitary wave computations, and many others. Such methods deliver superior performance while maintaining easy implementation. These numerical methods provide powerful tools for nonlinear wave studies.

Experiments have been closely associated with nonlinear wave studies since the early days of Russell's observation in 1834. In the 1960s and 1970s, much progress on nonlinear waves was made in the context of water waves. Since 1980s, optical experiments have become very active due to technological advances and exciting commercial applications (such as fiber communications). The high sophistication of optical experiments makes it possible to compare experimental results with nonlinear wave theories not only qualitatively but also quantitatively. This has become one of the main driving forces behind nonlinear wave studies. In recent years, optical experiments in periodic or quasi-periodic media (such as photonic crystal fibers and optically induced photonic lattices) have been attracting a lot of interest, and this motivates many theoretical investigations in this area. Another different but closely related physical subject is the Bose–Einstein condensates. The nonlinear atom-atom interaction in Bose–Einstein condensates gives rise to collective wave behaviors which are intimately related to nonlinear optical phenomena. The first experimental observation of Bose–Einstein condensates in 1995 and the subsequent award of Nobel prize to its observers in 2001 greatly stimulated theoretical and experimental research in this area. The dynamic interplay between theory and experiments in these physical subjects has been a signature of current nonlinear wave research, and this trend will continue in the days to come.

The aim of this book is to cover nonlinear waves from integrable to nonintegrable equations, from analysis to numerics, and from theory to experiments. These different aspects of nonlinear waves, which in the past have generally been covered in separate books, are combined in this single book. Thus this book is not only valuable for preparing graduate students of applied mathematics to enter the nonlinear wave field, but also useful for scientists in other disciplines (such as optics, fluid mechanics, and Bose–Einstein condensates) to learn how recent progress in the nonlinear wave theory can help them in their own research. Most materials in this book are self-contained, with detailed calculations carefully explained. Thus the reader should be able to learn these materials from this book without much help from other sources. Although we often use familiar wave equations to develop the analysis, the treatments we present are largely general and can be readily extended to other related equations.

The first chapter derives various nonlinear wave equations from generic model equations or from concrete physical systems. These wave equations will be heavily studied in later chapters. The second chapter develops the integrable theory for the NLS equation using the Riemann–Hilbert formulation. Various aspects of the integrable theory, such as the inverse scattering method, infinite conservation laws, the integrable hierarchy, and the connection between squared eigenfunctions and the linearized integrable equations, will be presented. The third chapter extends the integrable theory of the second chapter to higher orders (such as the vector NLS systems). Much of this extension is straightforward, thus the universal aspects of the integrable theory in Chapter 2 become evident. But some new features do appear in higher-order systems. Such new features will be highlighted in this chapter. The fourth chapter develops the soliton perturbation theory for weakly perturbed integrable equations, and then applies it to study various physical problems. This soliton perturbation theory is based on multiscale perturbation expansions, and its key component is to directly solve the linearized integrable equations using squared eigenfunctions. This perturbation theory not only can yield dynamical equations for soliton parameters, but can also obtain radiation fields. It will be shown that the calculation of radiation is critical for certain types of perturbed solitons (such as embedded solitons). The fifth chapter covers various analytical theories for nonintegrable equations. Topics in this chapter include the Vakhitov–Kolokolov stability criterion and its generalizations, the exponential asymptotics technique for nonlocal waves, dynamical theories for embedded solitons, theoretical analysis of fractal scatterings in solitary wave interactions, transverse stability analysis of solitary waves, and theories for wave collapse in multidimensions. These theories highlight the drastic differences between solution dynamics in integrable and nonintegrable equations. The sixth chapter covers a particular but important branch of nonintegrable theories, which is the nonlinear wave dynamics in periodic media. This subject exhibits rich and interesting phenomena which have no counterpart in homogeneous media. Both the analytical theories and experiments in optics and Bose–Einstein condensates will be presented, and the interplay between theory and experiment will be demonstrated. The last chapter presents miscellaneous numerical methods for various aspects of nonlinear wave computations. Simple but efficient numerical schemes for evolution simulations, computations of solitary waves and their stability spectra will be developed. The accuracy, numerical stability, and convergence rates of these numerical schemes will also be analyzed. In addition, sample MATLAB codes for all these numerical methods will be displayed.

It should be mentioned that the field of nonlinear waves has become huge. Numerous directions have been pursued, and rich results have been obtained. The topics covered in this book only represent a small fraction of this vast field. Many interesting results and methods have been left out. For instance, on the theories of integrable equations, we have completely omitted the Hirota method, the Hamiltonian and bi-Hamiltonian structures, Darboux and Bäcklund transformations, the Painlevé properties, inverse scattering in multidimensions, and so on. Other omitted topics include semiclassical limits of integrable equations, the normal-form perturbation theory for integrable equations, analysis of discrete systems, etc. In addition, an important area of application for nonlinear waves, namely nonlinear water waves, is not covered either. Nonetheless, the topics we do cover in this book form a coherent and self-contained body of knowledge on nonlinear waves. More importantly, many of these covered topics witnessed remarkable growth in recent years, and they could show further new developments in the near future.

The only software used in this book is MATLAB. In the last chapter on numerical methods, sample MATLAB codes are included for all numerical schemes which are presented, so that readers can directly use them or modify them to their purposes. These codes can also be found on the SIAM Web page associated with this book, www.siam.org/books/mm16, or on the author's Web page, www.cems.uvm.edu/~jyang. These MATLAB codes can be readily converted into other software such as *Mathematica* or C if the reader prefers.

This book can be used as a textbook for a graduate-level course on nonlinear waves. It can also be used as a reference book for researchers working in various fields such as applied mathematics, physics, and engineering, where nonlinear wave phenomenon is a common theme.

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