Preface

A model is a rendition of a reality which is often too difficult or impossible to handle directly. A model is always a simplification, and a good model is one that captures the essential features of reality, leaving the unessential out. Therefore, a good model helps us understand the interconnections between assumptions and different observations. A model need not be a mathematical model: Clothing designers, architectural firms, and engineers, for example, routinely test ideas with models which are sketchy renditions of how the final product would look or behave once realized. In medical research a model may be a mouse, where hypothetical human conditions are reproduced and new therapies are tested. As informative and useful as the results of an investigation with a model may be, they should always be taken with the caveat that a model is, after all, only a model.

Mathematical modeling is all about expressing features of interest of a real phenomenon in mathematical terms, with the implicit understanding that some of the questions of interest may be answered more accurately and definitively using mathematical tools. Physics is a success story of mathematical modeling, at times so successful with its high predictive power that we have the tendency to forget that "Laws of Nature," as expressed in mathematical terminology, are not reality itself but just models. In the past few decades, life sciences have been moving more and more toward the center stage of mathematical modeling, and the daunting complexity of even the simplest questions involving living organisms forces us to rethink the whole modeling paradigm. While it may be argued that mathematics is an axiomatic set of rules to handle content-free variables, and that its power comes from this very universality, yielding results that hold for any content, the design of a mathematical model must take both the underlying content and the scope of the investigation into account. Unlike Laws of Nature, the modeling paradigm is closer to what has been commonly admitted in statistics and nicely captured by the famous citation by George E. P. Box when discussing science and statistics: "All models are wrong but some are useful."

Mathematical modeling is a translational process, in which the problem of interest, described originally in nonmathematical natural language and in qualitative terms, is rendered in a quantitative mathematical form. It is therefore the first crucial step in the wider context of applied mathematics. The commitment to a particular mathematical formulation is decisive, determining what kind of mathematical tools will be needed, whether a reasonable solution can be found, and, in the affirmative case, how accurate a solution can be expected. Interesting real-world problems tend to be complex, and they can usually be studied at different scales, depending on the questions that we are trying to address and on the type of observations that are available. The scale at which the investigation is carried out is one of the factors determining the type of mathematics that most appropriately describes the problem. For this reason, the same phenomenon may be represented by a system of coupled differential equations as a large-scale mean phenomenon, or by a stochastic process, or a system of interacting stochastic processes, if describing it on a microscopic level. Occasionally, it may also be of interest to assess how the parameters which specify the model at one scale can be mapped into the parameters which describe it at a different scale. Understanding the connections between different scales brings enormous insight into the problem, and gives a rich toolbox for versatile research. The use of different scales also opens a world of interesting and challenging research problems for applied mathematicians.

This book draws a path in the mathematical modeling forest where we visit a number of typical problems arising in applications and derive suitable mathematical formulations using different scales. The point of view in this book is very practical, and therefore all of our modeling effort is shaped by considerations about the numerical simulations which will follow.

The mathematical tools that we use range from differential equations, ordinary and partial, to applied probability. While a certain level of mathematical fluency is assumed, some topics which have the tendency to fall through the cracks and to be left out from the standard calculus and differential equations courses are included in the treatment. Likewise, while scientific computing is not the focus of this book, each modeling paradigm is accompanied by a discussion of how it can be implemented numerically to check that the predictions of the models are in agreement with the understanding of the underlying program. Segments of MATLAB code illustrating how to complete the journey all the way to a computational model are included in most chapters, and several of the homework problems include a computational component.

This book was based on the idea that a modeler should be able to readily design mathematical models of basic phenomena which can then be combined in a model of much higher complexity. In real applications it is quite rare that the first model found is able to answer the posed questions in an entirely satisfactory manner: More realistically, most applications require the design of a sequence of models, usually of increasing complexity, with each subsequent model answering more accurately the initial question and any additional ones which may have been raised in the interim.

Mathematical modeling is an ill-defined field in the sense that no uniform modeling theory or paradigm exists. As the saying goes, the proof of the pudding is in its eating. Reflecting this plurality of modeling, each chapter introduces a different modeling style and illustrates with examples what is suitably modeled with this technique. Some small-scale phenomena require mathematical expressions which can deal with counting data of individual events, while at a larger scale the effect of an ensemble of small-scale phenomena may be appropriately described with differential equations, which capture the averaged effects instead of the individual events. A probabilistic framework is introduced when random events play an important role in the model, and is extensively used to capture salient features of noise, whose nature is inherently stochastic. Attention is constantly paid to the computer implementation of these different models throughout the book. Therefore, as the tools of probability are introduced and utilized in the design of the models, the basic notions of how to address them in a computational context are also presented.

The book is organized into nine chapters. The first four chapters are concerned with deterministic models, while the models introduced in the last five chapters all have stochastic elements. Here is a very brief overview of each chapter. Chapter 1 is a very concise and condensed review of calculus and differential equations, with the essential concepts and results that are extensively used throughout the other chapters. This chapter is not crucial for the rest of the book, and may be skipped and consulted selectively later if needed. The actual modeling begins in Chapter 2, which is dedicated to simple compartment models. Compartment models comprise interacting input-output units or compartments, and within each compartment mutually interacting sub-compartments may be specified. When zooming into the compartments, it may happen that the increased internal complexity, such as spatial distribution or age distribution, is best described in terms of a continuum model, governed by partial differential or integro-differential equations. This refinement process is described in the light of examples in Chapter 3.

It is not uncommon in engineering sciences to build prototypes that simulate the real setup of interest on a small laboratory scale. The physical phenomena are usually the same, thus the mathematical formalism applies both to the prototype and the true object, but the scale is different. Therefore, an important issue that a modeler needs to carefully address is robustness with respect to scaling. Scaling of models is analyzed in Chapter 4, which also deals with dimensional analysis, an equally fundamental concept which extracts the true degrees of freedom of a model and frees it from the slavery of units.

Stochastic models make a big entrance in Chapter 5, with the introduction of basic concepts such as probability distributions and methods to draw random realizations from given distributions, a crucial step in stochastic simulation. The important concept of noise is discussed in Chapter 6. Chapter 7 introduces waiting processes, which are central in modeling, e.g., biological phenomena, chemical kinetics, or light propagation in scattering media, as demonstrated by the examples in the chapter. Waiting processes lead naturally to the concept of Markov processes, which are more systematically introduced in Chapter 8. Finally, as spatial distribution is introduced, the modeler needs to take into account the mutual interaction of Markov processes. This interaction leads to the versatile concept of agent-based stochastic modeling and stochastic cellular automata, discussed in Chapter 9.

Each chapter ends with a small section suggesting references for further reading. Some of the references contain background information for readers who want to put the discussion on a more rigorous footing, while other references are meant to give either a historical perspective, or simply point toward other exciting application areas in which the discussed methodology is successfully applied. Finally, each chapter contains a number of exercises.

This book grew out of a one semester introductory graduate course on mathematical modeling taught at Case Western Reserve University. The authors wish

to express their gratitude to the splendid Case graduate and upper undergraduate students who took the class for all the valuable feedback and comments. We are particularly thankful to Andrea Arnold for carefully reading and commenting on the manuscript. Also, we wish to thank the anonymous referees for suggestions and comments.