Preface

Measurable dynamics has traditionally referred to ergodic theory, which is in some sense a sister topic to dynamical systems and chaos theory. However, the topic has until recently been a highly theoretical mathematical topic which is generally less obvious to those practitioners in applied areas, who may not find obvious links to practical, real-world problems. During the past decade, facilitated by the advent of high-speed computers, it has become practical to represent the notion of a transfer operator discretely but to high resolution thanks to rapidly developing algorithms and new numerical methods designed for the purpose. An early book on this general topic is *Cell-to-Cell Mapping: A Method of Global Analysis for Nonlinear Systems* [167] from 1987.[1] A tremendous amount of progress and sophistication has come to the empirical perspective since then.

Rather than discussing the behaviors of complex dynamical systems in terms of following the fate of single trajectories, it is now possible to empirically discuss global questions in terms of evolution of density. Now complementary to the traditional geometric methods of dynamical systems transport study, particularly by stable and unstable manifold structure and bifurcation analysis, we can analyze transport activity and evolution by matrix representation of the Frobenius–Perron transfer operator. While the traditional methods allow for an analytic approach, when they work, the new and fast-developing computational tools discussed here allow for detailed analysis of real-world problems that are simply beyond the reach of traditional methods. Here we will draw connections between the new methods of transport analysis based on transfer operators and the more traditional methods. The goal of this book is not to become a presentation of the general topic of dynamical systems, as there are already several excellent textbooks that achieve this goal in a manner better than we can hope. We will bring together several areas, as we will draw connections between topological dynamics, symbolic dynamics, and information theory to show that they are also highly relevant to the Ulam–Galerkin representations. In these parts of the discussion, we will compare and contrast notions from topological dynamics to measurable dynamics, the latter being the first topic of this book. That is, if measurable dynamics means a discussion of a dynamical system in consideration of how much, how big, and other notions that require measure structure to discuss transport rates, topological dynamics can be considered as a parallel topic of study that asks similar questions in the absence of a measure that begets scale. As such, the mechanism and geometry of transport are more the focus. Therefore, including a discussion of topological dynamics in our primary discussion here on measurable dynamics should be considered complementary.

[1] Recent terminology has come to call these "set oriented" methods.

There are several excellent previous related texts on mathematical aspects of transfer operators which we wish to recommend as possible supplements. In particular, Lasota and Mackay [198] give a highly regarded discussion of the theoretical perspective of Frobenius–Perron operators in dynamical systems, whose material we overlap in as far as we need these elements for the computational discussion here. Boyarsky and Gora [50] also give a sharp presentation of an ensembles density perspective in dynamical systems, but more specialized for one-dimensional maps, and some of the material and proofs therein are difficult to find elsewhere. Of course the book by Baladi [11] is important in that it gives a thoroughly rigorous presentation of transfer operators, including a unique perspective. We recommend highly the book by Zhou and Ding, [324], which covers a great deal of theoretical information complementary to the work discussed in this book, including Ulam's method and piecewise constant approximations of invariant density, piecewise linear Markov models, and especially analysis of convergence. Also an in-depth study can be found concerning connections of the theory of Frobenius–Perron operators and the adjoint Koopman operator, as well as useful background in measure theory and functional analysis. The book by McCauley [215] includes a useful perspective regarding what is becoming a modern perspective on computational insight into behaviors of dynamical systems, especially experimentally observed dynamical systems. That is, finite realizations of chaotic data can give a great deal of insight. This is a major theme which we also develop here toward the perspective that a finite time sample of a dynamical system is not just an estimate of the long time behavior, as suggested perhaps by the traditional perspective, but in fact finite time samples are most useful in their own right toward understanding finite time behavior of a dynamical system. After all, any practical, real-world observation of a dynamical system can be argued to exist only during a time window which cannot possibly be infinite in duration.

There are many excellent textbooks on the general theory of dynamical systems, clearly including Robinson [268], Guckenheimer and Holmes [146], Devaney [95], Alligood, Sauer, and Yorke [2], Strogatz [301], Perko [251], Meiss [218], Ott [244], Arnold [4], Wiggins [316], and Melo and van Strein [89], to name a few. Each of these has been very popular and successful, and each is particularly strong in special aspects of dynamical systems as well as broad presentation. We cannot and should not hope to repeat these works in this presentation, but we do give what we hope is enough background of the general dynamical systems theory in order that this work can be somewhat self-contained for the nonspecialist. Therefore, there is some overlap with other texts insofar as background information on the general theory is given, and we encourage the reader to investigate further in some of the other cited texts for more depth and other perspectives. More to the point of the central theme of this textbook, the review article by Dellnitz and Junge [87] and then later the Ph.D. thesis by Padberg [247] (advised by Dellnitz) both give excellent presentations of a more computationally based perspective of measurable dynamical systems in common with the present text, and we highly recommend them. A summary of the German school's approach to the empirical study of dynamical systems can be found in [112], and [82]. Also, we recommend the review by Froyland [121]. Finally, we highly recommend the book by Hsu [167], and see also [166], which is an early and less often cited work in the current literature, as we rarely see "cell-to-cell mappings" cited lately. While lacking the transfer oriented formalism behind the analysis, this cell-to-cell mapping paradigm is clearly a precursor to the computational methods which are now commonly called set oriented methods. Also, we include a discussion and contrast to the early ideas by Ulam [307]

called the Ulam method. Here we hope to give a useful broad presentation in a manner that includes some necessary background to allow a sophisticated but otherwise not specialized student or researcher to dive into this topic.

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