Preface

This book explores how to derive relatively simple dynamical equations that model complex physical interactions. The book arises out of the growing interest in and applications of modern dynamical systems theory. Due to my background, fluid flows and other continuum dynamics form many of the applications we investigate. The triple aim of the book is to use sound theory to explore algebraic techniques, develop interesting applications, and discover general modeling principles.

Mentor and colleague Prof. E. O. Tuck was discussing the undergraduate curriculum with another who was espousing the need for courses on elementary mathematics with an advanced approach. Prof. Tuck's riposte was "but that is exactly the opposite of what I want; I want advanced mathematics from an elementary approach." Similarly, this book aims to develop advanced mathematical modeling methods and discuss their subtleties with as elementary mathematics as possible. The assumed background knowledge is common undergraduate linear algebra, calculus, and differential equations, but there is no need for functional analysis, advanced differential geometry, or even complex analysis.

The basis for the methodology is both the theory and the geometric picture of both coordinate transforms and invariant manifolds in dynamical systems: in particular, we heavily use center and slow manifolds. The wonderful aspect of this approach is the range of geometric interpretations of the modeling process. Simple geometric pictures inspire sound methods of analysis and construction. Further, the pictures that we draw of state spaces also provide a route for better assessing limitations and strengths in a model. Geometry and algebra form a powerful partnership.

> … duality between algebra and geometry was discovered by René Descartes: every geometric object has an algebraic description, every algebraic formula determines a geometric object. Humans tend to use the algebraic version for calculation, and the geometric one for imagination.
>
> Fearful symmetry, *Stewart and Golubitsky.*

The theme of this book is that coordinate transforms and center manifolds provide a powerfully enhanced and unified view of a swath of other complex system modeling methodologies, such as averaging, homogenization, multiple scales, singular perturbations, two timing, and WKB theory.

One main reason complex systems are complex is that there are many interacting components. We generally posit some network of interacting "agents" or "particles." The simplest such complex networks arise through the highly regular structure of space where nearest neighbor interactions dominate. The domain of spatiotemporal dynamics provides us with many examples. Under time evolution, coherent patterns (stripes on a tiger) or incoherent patterns (turbulence) emerge over time. We seek to find ways

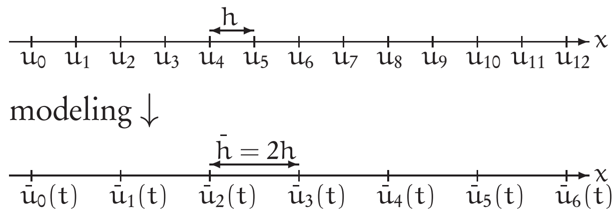

Figure 1. *Our modeling analyzes the dynamics* (0.1) *of* $u_i(t)$ *on the (upper) fine-scale lattice, spacing* h, *and systematically maps it into the dynamics* (0.2) *of* $\bar{u}_j(t)$ *on the (lower) coarse grid, spacing* $2h$.

to characterize, to model, the coherent or incoherent behavior that we see. What is the aggregate behavior? How can the whole appear to be more than the sum of its parts?

Example 0.1 (dynamics on a 1D lattice). Let us loosely overview one application in multiscale modeling. Distribute "particles" at the grid points of a 1D lattice (with spacing h, say), as shown in the upper part of Figure 1. Each of these particles has some property called u, perhaps temperature, that evolves in time so the ith particle has property $u_i(t)$. For simplicity, restrict attention to linear dynamics. Then the generic, spatially discrete, only nearest neighbor interactions system that preserves total u-stuff is the advection-diffusion equations

$$\frac{du_i}{dt} = -c\tfrac{1}{2}(u_{i+1} - u_{i-1}) + d(u_{i+1} - 2u_i + u_{i-1}) \tag{0.1}$$

for some constants c and d. This equation moves u-stuff around with "advection speed" ch and spreads u-stuff with "diffusion" dh^2.

Suppose we seek to model the dynamics (0.1) on a grid coarser by a factor of two, spacing $\bar{h} = 2h$, as in the lower part of Figure 1. Grid point j on the coarser grid would correspond to fine grid point $i = 2j$. Let $\bar{u}_j(t)$ be the coarse grid values of the u-stuff. Techniques we develop (Roberts, 2009c, e.g.) justifiably model the fine-grid dynamics by the coarse-grid equation

$$\begin{aligned} \frac{d\bar{u}_j}{dt} &\approx -\bar{c}\tfrac{1}{2}(\bar{u}_{j+1} - \bar{u}_{j-1}) + \bar{d}(\bar{u}_{j+1} - 2\bar{u}_j + \bar{u}_{j-1}), \\ \text{where} \quad \bar{c} &= \tfrac{1}{2}c \quad \text{and} \quad \bar{d} = \tfrac{1}{4}d + \frac{c^2}{16d}\,. \end{aligned} \tag{0.2}$$

This is another advection-diffusion equation for the movement and spread of u-stuff but with appropriately renormalized coefficients to suit the coarser grid.

The renormalization of the coefficients in (0.2) has components $\frac{1}{2}c$ and $\frac{1}{4}d$ that all traditional linear modeling techniques would derive. Our more careful techniques show that the coarse "diffusion" should be enhanced in proportion to c^2/d. This last correction helps ensure that the coarse-scale model preserves stability. It arises through more carefully resolving the consequences of the microscale dynamics on the fine grid.

Having generated a mapping of dynamics from one grid to a coarser grid, we may repeat the mapping across a whole hierarchy of grids, as indicated schematically by Figure 2. At each level of the grid the dynamics would be an advection-diffusion

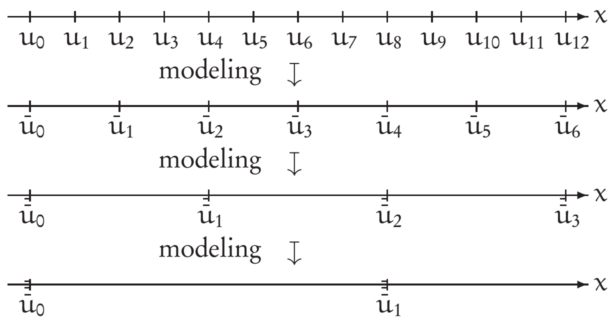

Figure 2. *The modeling* (0.1)$\mapsto$(0.2) *from a fine lattice to a coarser lattice may be repeated across a hierarchy of lattices to understand dynamics on many length scales.*

> equation. Although each of the models (0.1) and (0.2) expresses simple linear dynamics, the transformation of the model from one grid scale to another is *nonlinear* through the nonlinear dependence of $\bar{d}$ upon c and d. Repeating the nonlinear transformation (0.1)$\mapsto$(0.2) across many grid scales morphs diffusion-dominated microscale dynamics into an upwind, advection-dominated, macroscale model. Analogous nonlinear transformations of other systems have the potential to demonstrate and illuminate emergence of qualitatively new dynamical rules on different scales of a complex system.

Example 0.1 links many mathematical and computational techniques, such as multigrid algorithms, wavelets, cellular automata, lattice dynamics, and renormalization. Our main tool to tease out such emergent dynamics will be judicious coordinate transforms and their analogues in invariant manifolds. Where possible our analysis will be rigorous. However, rigor rapidly disintegrates in the face of practical realities: in many of the applications discussed, rigorous support is lacking, but let's not let that lack prevent us from making progress in understanding the modeling of emergent dynamics.

> Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where explorers often get lost. Rigour should be a signal to the historian that the maps have been made, and the real explorers have gone elsewhere. *W.S. Anglin*

I emphasize that modeling from one scale to another is a nonlinear process. Example 0.1 shows this nonlinearity in the nonlinear mapping of coefficients from one scale to the next. Why emphasize this nonlinearity? Because most people mostly use linear arguments in modeling. For such linear derivations the dynamics on one scale looks much the same as that on another, due to the linearity; thus they view the whole as always the sum of its parts. But because we here recognize the inherent nonlinearity in modeling, we see that the whole is more than the sum of its parts.[1]

Another major reason complex systems are complex is the interaction of many physical processes: "For example, a complete computational model of a large-scale fusion device

[1] Huge discussions take place over "emergence" and "strong emergence" in complex systems. The mapping of systems from microscale to macroscale, being nonlinear, need not be uniquely invertible. Because we recognize this possibility of multiple causations, the nonlinear nature of modeling from one scale to another supports the "strong emergence" tenet that macroscale phenomena need not be always traceable to the microscale.

is a complex system involving issues of fluid dynamics, deformation of solid materials, thermal effects, ablation, fracture, corrosion and aging of materials, radiation and many other phenomena" (Brown et al., 2008, §2.1.1, e.g.). Most systematic approaches to modeling require multiphysical processes to all interact at the same "order" in the modeling. In contrast, techniques developed herein flexibly allow different physical effects to be of different orders of magnitudes, even in different locations in space-time.

Example 0.2 (thin fluid flow on a fiber). Even the relatively simple flow of a thin film of fluid along a cylindrical fiber is physically complicated (Roberts and Li, 2006, e.g.). Gravity drains the fluid along the fiber but also pulls fluid tangentially around the fiber in some places and normally develops a "hydrostatic" pressure in other places with gradients that drive the flow in other directions. Surface tension then tends to form drops of shape induced by curvature of the cylindrical fiber and the curvature of the drop. The flow of the fluid in the drops is then affected by inertia, whereas the thinner fluid outside the drops is little influenced by inertia. All these processes occur in the flow, with different balances occurring in different locations and times. Sound modeling has to cope with such multiphysics.

Find more information Developments and applications further to those developed herein are documented on my web page for this book.[2] My website also provides services to construct slow and center manifold models of autonomous or nonautonomous differential equation systems that you might enter.

[2] `http://www.siam.org/books/mm20`