Preface to the Revised Edition

For this revised edition, I have corrected many typographical errors and improved a number of figures (3.2, 3.8, 4.1-3, 4.15, 5.7, 8.3, 8.5, 8.6, 8.7,8.14, 8.19, 9.7). The exposition in Section 3.2 has been reordered and made more complete, and I have added a couple of new of theorems (e.g., Theorem 7.3, Birkhoff Transitivity and Theorem 3.5, Completeness of C^0). Thanks to the sharp eyes and careful thinking of a number of readers, the statements and/or proofs of a number of other theorems have been improved (e.g., 3.24, 4.6, 4.8, 4.42, 4.23, 4.46, 5.9, 5.10, 5.11, and 7.12). Finally there are are several new exercises (3.3, 3.5, 4.16 and 7.9).

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Preface

On one level, this text can be viewed as suitable for a traditional course on ordinary differential equations (ODEs). Since differential equations are the basis for models of any physical systems that exhibit smooth change, students in all areas of the mathematical sciences and engineering require the tools to understand the methods for solving these equations. It is traditional for this exposure to start during the second year of training in calculus, where the basic methods of solving one- and two-dimensional (primarily linear) ODEs are studied. The typical reader of this text will have had such a course, as well as an introduction to analysis where the theoretical foundations (the ε's and δ's) of calculus are elucidated. The material for this text has been developed over a decade in a course given to upper-division undergraduates and beginning graduate students in applied mathematics, engineering, and physics at the University of Colorado. In a one-semester course, I typically cover most of the material in Chapters 1–6 and add a selection of sections from later chapters.

There are a number of classic texts for a traditional differential equations course, for example (Coddington and Levinson 1955; Hirsch and Smale 1974; Hartman 2002). Such courses usually begin with a study of linear systems; we begin there as well in Chapter 2. Matrix algebra is fundamental to this treatment, so we give a brief discussion of eigenvector methods and an extensive treatment of the matrix exponential. The next stage in the traditional course is to provide a foundation for the study of nonlinear differential equations by showing that, under certain conditions, these equations have solutions (existence) and that there is only one solution that satisfies a given initial condition (uniqueness). The theoretical underpinning of this result, as well as many other results in applied mathematics, is the majestic contraction mapping theorem. Chapter 3 provides a self-contained introduction to the analytic foundations needed to understand this theorem. Once this tool is concretely understood, students see that many proofs quickly yield to its power. It is possible to omit §§3.3–3.5, as most of the material is not heavily used in later chapters, although at least passing acquaintance with Theorem 3.19 and Lemma 3.28 (Grönwall) is to be encouraged.

However, this text does not aim to cover only the material in such a traditional ODE course; rather, it aspires to serve as an introduction to the more modern theory of dynamical systems. The emphasis is on obtaining a *qualitative* understanding of the properties of *differential* dynamical systems, namely, those evolution rules that describe smooth evolution in time.[1] The primary concept of this study, the *flow*, is introduced in Chapter 4. The qualitative theory is often concerned with questions of shape and asymptotic behavior that lead us to use topological notions such as conjugacy in the classification of dynamics.

[1]This is not to say that the dynamical systems that we study are always *differentiable*—vector fields need not be smooth.

The classification of dynamical behavior begins with the simplest orbits, equilibria and periodic orbits. As Henri Poincaré noted in his classic *New Methods in Celestial Mechanics*, (1892, Vol. 1, §36),

> *what renders these periodic solutions so precious to us is that they are, so to speak, the only breach through which we may attempt to penetrate an area hitherto deemed inaccessible.*

Only in the demonstration that dynamics in the neighborhood of some of these orbits is conjugate to their linearization is it seen that the predisposition of applied scientists to concentrate on linear systems has any value whatsoever.

The local classification of equilibria leads to the theory of invariant manifolds in Chapter 5. The stable and unstable manifolds, proved to exist for a hyperbolic saddle, give rise to one prominent mechanism for chaos—heteroclinic intersection. The center manifold theorem is also important preparation for the treatment of bifurcations in Chapter 8.

As mathematicians, allow yourselves to become entranced by the exceptions to the validity of linearization, namely, with those orbits that are nonhyperbolic. It is in the study of these exceptions that we find the most beautiful dynamics—even in the case of the phase plane, to which we return in Chapter 6. The first three sections of this chapter are fundamental; §§6.4–6.8 can be omitted in favor of later chapters. As we see in Chapter 8, the exceptional cases form the organizing centers for the behavior of systems undergoing changing parameters. A qualitative change in behavior under a small change of parameters is called a bifurcation. A complete exegesis of theory of bifurcations requires a full text on its own, and there are many excellent texts appropriate for a more advanced class (Guckenheimer and Holmes 1983; Golubitsky and Schaeffer 1985; Kuznetsov 1995). We introduce the reader to the basic ideas of normal forms and treat codimension-one and -two bifurcations.

Perhaps the most exciting recent developments in dynamical systems are those that show that even simple systems can behave in complicated ways, namely, the phenomena of *chaos*. In Chapter 7, we introduce the reader to the concepts necessary for understanding chaos: Lyapunov exponents, transitivity, fractals, etc. We also give an extensive discussion of Melnikov's method for the onset of chaos in Chapter 8. A more advanced treatment of chaotic dynamics requires a discussion of discrete dynamics (mappings) and can be found in texts such as (Katok and Hasselblatt 1999; Robinson 1999; Wiggins 2003).

The final chapter treats the subject closest to this author's heart: Hamiltonian dynamics. Since the basic models of physics all have a Hamiltonian (or Lagrangian) formulation, it is worthwhile to become familiar with them. While a traditional physics text treats these on a concrete level, this book provides an introduction to some of the geometrical aspects of Hamiltonian dynamics, including a discussion of their variational foundation, spectral properties, the KAM theorem, and transition to chaos. Again, there are several advanced texts that go much further, for example (Arnold 1978; Lichtenberg and Lieberman 1992; Meyer and Hall 1992).

While the proofs of many of the classical theorems are included, this text is not just an abstract treatment of ODEs but an attempt to place the theory in the context of its many applications to physics, biology, chemistry, and engineering. Examples in such areas as population modeling, fluid convection, electronics, and mechanics are discussed throughout the text, and especially in Chapter 1. The exercises introduce the reader to many more. Furthermore, to develop a geometrical understanding of dynamics, each student must experiment; we provide some examples of simple codes written in Maple, Mathematica, and MATLAB in the appendix, and we use the exercises to encourage the student to explore further. There are several texts that focus completely on using one or more of tools like these to explore dynamics (Lynch 2001; Baumann 2004).

I hope that this book conveys a bit of my amazement with the beauty and utility of this field. Dynamical systems is the perfect combination of analysis, geometry, and physical intuition. Central questions in dynamics have been formulated for centuries, and although some have been solved in the past few years, many await solution by the next generation.

> *It is far better to foresee even without certainty than not to foresee at all.*
> (Henri Poincaré, *The Foundations of Science*)

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