Date: October 18, 2012

Title: Introduction to optimization and semi-differential calculus

Subject: Comments, corrections, typos

Chapter 1

Section 4.3.1 on page 9.

Convention 4.1 should be dropped. It will not be respected in Chapter 2.

Addition at the end of page 10.

For completeness add the following definition and remark.

Although it is simpler to work with the definition of continuity for functions \( f : \mathbb{R}^n \to \mathbb{R} \), existence theorems in the next chapter will only necessitate the \textit{weaker} notion of continuity of a function \( f : U \to \mathbb{R}^m \) on a closed subset \( U \) endowed with the relative topology.

\textbf{Definition 4.12.} Given \( U, \emptyset \neq U \subset \mathbb{R}^n \), and a function \( f : U \to \mathbb{R}^m \) for two integers \( n \geq 1 \) and \( m \geq 1 \). The function \( f \) is \textit{continuous} at \( x \in U \) if \( \forall \varepsilon > 0, \exists \delta(x) > 0 \) such that

\[
\forall y \in U \text{ tel que } \|y - x\|_{\mathbb{R}^n} < \delta(x), \quad \|f(y) - f(x)\|_{\mathbb{R}^m} < \varepsilon.
\]

The function \( f \) is \textit{continuous on} \( U \) if \( f \) is continuous at all points of \( U \).

\textbf{Remark 4.3.} When \( U \) is a closed subset of \( \mathbb{R}^n \), there exists a \textit{continuous extension} \( \hat{f} : \mathbb{R}^n \to \mathbb{R}^m \) of \( f \) from \( U \) to the whole space \( \mathbb{R}^n \) which is continuous on \( U \) in the sense of Definition 4.11 (see, for instance, W. Rudin [1, Ex. 5, p. 91]).

Chapter 2

Section 3, Definition 3.1 on page 13.

This definition is not complete and can be simplified as follows.

\textbf{Definition 3.1.} Let \( f : \mathbb{R}^n \to \mathbb{R} \) and \( U \subset \mathbb{R}^n \).

(i) Associate with \( f \) its \textit{effective domain}

\[
\text{dom } f \overset{\text{def}}{=} \{ x \in \mathbb{R}^n : -\infty < f(x) < +\infty \}.
\]

It will also be simply referred to as the \textit{domain} of \( f \).
(ii) The infimum of \( f \) with respect to \( U \) is defined as follows:
\[
\inf f(U) \overset{\text{def}}{=} \begin{cases} 
\inf f(U \cap \text{dom } f), & \text{if } f(U) \subset \mathbb{R} \cup \{+\infty\}, \\
-\infty, & \text{if } \exists x \in U \text{ such that } f(x) = -\infty.
\end{cases}
\]

We shall also use the notation \( \inf_{x \in U} f(x) \).

The supremum of \( f \) with respect to \( U \) is defined as follows:
\[
\sup f(U) \overset{\text{def}}{=} \begin{cases} 
\sup f(U \cap \text{dom } f), & \text{if } f(U) \subset \mathbb{R} \cup \{-\infty\}, \\
+\infty, & \text{if } \exists x \in U \text{ such that } f(x) = +\infty.
\end{cases}
\]

We shall also use the notation \( \sup_{x \in U} f(x) \).

Infima and suprema constitute the set of extrema of \( f \) in \( U \).

(iii) When there exists \( a \in U \) such that \( f(a) = \inf f(U) \), \( f \) is said to reach its minimum at a point of \( U \) and it is written as
\[
\min f(U) \text{ or } \min_{x \in U} f(x).
\]

The set of all minimizing points of \( f \) in \( U \) is denoted
\[
\arg\min f(U) \overset{\text{def}}{=} \{ a \in U : f(a) = \inf f(U) \}.
\]

When there exists \( b \in U \) such that \( f(b) = \sup f(U) \), \( f \) is said to reach its maximum at a point of \( U \) and it is written as
\[
\max f(U) \text{ or } \max_{x \in U} f(x).
\]

The set of all maximizing points of \( f \) in \( U \) is denoted
\[
\arg\max f(U) \overset{\text{def}}{=} \{ b \in U : f(b) = \sup f(U) \}.
\]

Section 3, Theorem 3.2 on page 15.

As a result of Definition 3.1, the proof of Theorem 3.2 has to be revisited as follows.

**Theorem 3.2.** Let \( U \subset \mathbb{R}^n \) and \( f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm \infty\} \). Then
\[
\inf f(U) = \inf f_U(U) = \inf f_U(\mathbb{R}^n) \quad \text{and} \quad \arg\min f(U) = U \cap \arg\min f_U(\mathbb{R}^n).
\]

If, in addition, \( \inf f(U) < +\infty \), then \( \arg\min f(U) = \arg\min f_U(\mathbb{R}^n) \).

**Proof.** \( \inf f(U) = \inf f_U(U) \) since \( f_U = f \) on \( U \). If there exists \( x \in U \) such that \( f(x) = -\infty \), then \( f_U(x) = -\infty \) and, by definition, \( \inf f_U(\mathbb{R}^n) = -\infty = \inf f(U) \). If \( f(U) \subset \mathbb{R} \cup \{+\infty\} \), then

\[
\inf f(U) = \inf f_U(U).
\]
\( f(U) = f_U(U) \subset f_U(\mathbb{R}^n) \subset \mathbb{R} \cup \{+\infty\} \) and \( \text{dom } f_U = U \cap \text{dom } f \) since \( f_U(x) = +\infty \) on \( \mathbb{R}^n \setminus U \) and \( f(x) > -\infty \) on \( U \). Therefore

\[
\inf f(U \cap \text{dom } f) = \inf f(\text{dom } f_U) = \inf f_U(\mathbb{R}^n \cap \text{dom } f_U)
\]

and, by definition of \( \inf f(U) \), \( \inf f_U(U) \) and \( \inf f_U(\mathbb{R}^n) \),

\[
\inf f(U) = \inf f_U(U) = \inf f_U(\mathbb{R}^n).
\]

As \( \inf f(U) = \inf f_U(U) = \inf f_U(\mathbb{R}^n) \) and \( U \subset \mathbb{R}^n \), we have
\[
\text{argmin } f(U) = \text{argmin } f_U(U) = \text{argmin } f_U(\mathbb{R}^n).
\]

◮ Proof of part (iii) of Theorem 7.1 on page 34. Change \( d_U(x) = -\infty \) for \( d_U(x) = +\infty \) on lines 9 and 3 from bottom:

(iii) By convention, \( U = \emptyset \) implies \( d_U(x) = +\infty \) that is convex, also by convention.

\[
\ldots
\]

Conversely, if \( d_U(x) = +\infty \) for some \( x \), then, by convention, \( U = \emptyset \) which is convex, also by convention.

◮ Section 7.3, Example 7.2 on page 41.

The parameter \( a \) should go to \( +\infty \) and not to 0.

“and \( \lim_{a \to +\infty} F(a, a + 1) = 0. \)”

Appendix B

◮ Proof of part (ii) of Exercise 10.6 on page 299.

(ii) We need to show that, at each point \( x \in \mathbb{R}^n \) and for each \( h < e^{f(x)} \), there exists a neighborhood \( V(x) \) of \( x \) such that for all \( y \in V(x) \), \( h < e^{f(y)} \). If \( h \leq 0 \), then for all \( x \in \mathbb{R}^n \), \( e^{f(x)} > 0 \geq h \) and the neighborhood can be chosen as \( V(x) = \mathbb{R}^n \). If \( h > 0 \), take the log of both side of the strict inequality. As the function log is monotone strictly increasing, it preserves the strict inequalities: \( \log h < f(x) \). As \( f \) is lsc at \( x \), there exists a neighborhood \( V(x) \) of \( x \) such that

\[
\forall y \in V(x), \quad \log h < f(y).
\]

Finally, as the exponential function \( t \mapsto e^t : \mathbb{R} \to \mathbb{R} \) is monotone strictly increasing, we get

\[
\forall y \in V(x), \quad h = e^{\log h} < e^{f(y)}
\]

and, by definition, \( e^{f(x)} \) is lsc at \( x \).