
Date: October 18, 2012

Title: Introduction to optimization and semi-differential calculus

Subject: Comments, corrections, typos

Chapter 1

▶ Section 4.3.1 on page 9.

Convention 4.1 should be dropped. It will not be respected in Chapter 2.

▶ Addition at the end of page 10.

For completeness add the following definition and remark.

Although it is simpler to work with the definition of continuity for functions $f : \mathbb{R}^n \to \mathbb{R}$, existence theorems in the next chapter will only necessitate the *weaker* notion of continuity of a function $f : U \to \mathbb{R}^m$ on a closed subset U endowed with the relative topology.

Definition 4.12. Given U, $\varnothing \neq U \subset \mathbb{R}^n$, and a function $f : U \to \mathbb{R}^m$ for two integers $n \geq 1$ and $m \geq 1$. The function f is *continuous* at $x \in U$ if $\forall \varepsilon > 0$, $\exists \delta(x) > 0$ such that

$$\boxed{\forall y \in U} \text{ tel que } \|y - x\|_{\mathbb{R}^n} < \delta(x), \quad \|f(y) - f(x)\|_{\mathbb{R}^m} < \varepsilon.$$

The function f is *continuous on* U if f is continuous at all points of U.

Remark 4.3. When U is a closed subset of $\mathbb{R}^n$, there exists a *continuous extension* $\hat{f} : \mathbb{R}^n \to \mathbb{R}^m$ of f from U to the whole space $\mathbb{R}^n$ which is continuous on U in the sense of Definition 4.11 (see, for instance, W. Rudin [1, Ex. 5, p. 91]).

Chapter 2

▶ Section 3, Definition 3.1 on page 13.

This definition is not complete and can be simplified as follows.

Definition 3.1. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $U \subset \mathbb{R}^n$.

(i) Associate with f its *effective domain*

$$\operatorname{dom} f \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : -\infty < f(x) < +\infty\}.$$

It will also be simply referred to as the *domain* of f.

(ii) The *infimum* of f with respect to U is defined as follows:

$$\inf f(U) \stackrel{\text{def}}{=} \begin{cases} \inf f(U \cap \text{dom } f), & \text{if } f(U) \subset \mathbb{R} \cup \{+\infty\}, \\ -\infty, & \text{if } \exists x \in U \text{ such that } f(x) = -\infty. \end{cases}$$

We shall also use the notation $\inf_{x \in U} f(x)$.

The *supremum* of f with respect to U is defined as follows:

$$\sup f(U) \stackrel{\text{def}}{=} \begin{cases} \sup f(U \cap \text{dom } f), & \text{if } f(U) \subset \mathbb{R} \cup \{-\infty\}, \\ +\infty, & \text{if } \exists x \in U \text{ such that } f(x) = +\infty. \end{cases}$$

We shall also use the notation $\sup_{x \in U} f(x)$.

Infima and suprema constitute the set of *extrema* of f in U.

(iii) When there exists $a \in U$ such that $f(a) = \inf f(U)$, f is said to reach its *minimum* at a point of U and it is written as

$$\min f(U) \text{ or } \min_{x \in U} f(x).$$

The *set of all minimizing points of f in U* is denoted

$$\text{argmin} f(U) \stackrel{\text{def}}{=} \{a \in U : f(a) = \inf f(U)\}.$$

When there exists $b \in U$ such that $f(b) = \sup f(U)$, f is said to reach its *maximum* at a point of U and it is written as

$$\max f(U) \text{ or } \max_{x \in U} f(x).$$

The *set of all maximizing points of f in U* is denoted

$$\text{argmax} f(U) \stackrel{\text{def}}{=} \{b \in U : f(b) = \sup f(U)\}.$$

► Section 3, Theorem 3.2 on page 15.

As a result of Definition 3.1, the proof of Theorem 3.2 has to be revisited as follows.

Theorem 3.2. Let $U \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$. Then

$$\inf f(U) = \inf f_U(U) = \inf f_U(\mathbb{R}^n) \quad \text{and} \quad \text{argmin} f(U) = U \cap \text{argmin} f_U(\mathbb{R}^n).$$

If, in addition, $\inf f(U) < +\infty$, then $\text{argmin} f(U) = \text{argmin} f_U(\mathbb{R}^n)$.

Proof. $\inf f(U) = \inf f_U(U)$ since $f_U = f$ on U. If there exists $x \in U$ such that $f(x) = -\infty$, then $f_U(x) = -\infty$ and, by definition,
$\inf f_U(\mathbb{R}^n) = -\infty = \inf f(U)$. If $f(U) \subset \mathbb{R} \cup \{+\infty\}$, then

$f(U) = f_U(U) \subset f_U(\mathbb{R}^n) \subset \mathbb{R} \cup \{+\infty\}$ and dom $f_U = U \cap$ dom f since $f_U(x) = +\infty$ on $\mathbb{R}^n \setminus U$ and $f(x) > -\infty$ on U. Therefore

$$\inf f(U \cap \text{dom } f) = \inf f(\text{dom } f_U) = \inf f_U(\text{dom } f_U) = \inf f_U(\mathbb{R}^n \cap \text{dom } f_U)$$

and, by definition of $\inf f(U)$, $\inf f_U(U)$ and $\inf f_U(\mathbb{R}^n)$,

$$\inf f(U) = \inf f_U(U) = \inf f_U(\mathbb{R}^n).$$

As $\inf f(U) = \inf f_U(U) = \inf f_U(\mathbb{R}^n)$ and $U \subset \mathbb{R}^n$, we have $\text{argmin} f(U) = \text{argmin} f_U(U) \subset \text{argmin} f_U(\mathbb{R}^n)$ and $\text{argmin} f_U(U) = U \cap \text{argmin} f_U(\mathbb{R}^n)$.. If, in addition, $\inf f(U) < +\infty$, then, by definition of f_U, $\text{argmin} f_U(\mathbb{R}^n) \subset \text{argmin} f_U(U) = \text{argmin} f(U)$. □

▶ Proof of part (iii) of Theorem 7.1 on page 34. Change $d_U(x) = -\infty$ for $d_U(x) = +\infty$ on lines 9 and 3 from bottom:

(iii) By convention, $U = \varnothing$ implies $d_U(x) = +\infty$ that is convex, also by convention.

...

Conversely, if $d_U(x) = +\infty$ for some x, then, by convention, $U = \varnothing$ which is convex, also by convention.

▶ Section 7.3, Example 7.2 on page 41.

The parameter a should go to $+\infty$ and not to 0.

"and $\lim_{a \to +\infty} F(a, a+1) = 0$."

Appendix B

▶ Proof of part (ii) of Exercise 10.6 on page 299.

(ii) We need to show that, at each point $x \in \mathbb{R}^n$ and for each $h < e^{f(x)}$, there exists a neighborhood $V(x)$ of x such that for all $y \in V(x)$, $h < e^{f(y)}$. If $h \leq 0$, then for all $x \in \mathbb{R}^n$, $e^{f(x)} > 0 \geq h$ and the neighborhood can be chosen as $V(x) = \mathbb{R}^n$. If $h > 0$, take the log of both side of the strict inequality. As the function log is monotone strictly increasing, it preserves the strict inequalities: $\log h < f(x)$. As f is lsc at x, there exists a neighborhood $V(x)$ of x such that

$$\forall y \in V(x), \quad \log h < f(y).$$

Finally, as the exponential function $t \mapsto e^t : \mathbb{R} \to \mathbb{R}$ is monotone strictly increasing, we get

$$\forall y \in V(x), \quad h = e^{\log h} < e^{f(y)}$$

and, by definition, $e^{f(x)}$ is lsc at x.
