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**Variational Analysis in Sobolev and *BV* Spaces. Applications to PDEs and Optimization. Second Edition.** *By Hedy Attouch, Giuseppe Buttazzo, and Gérard Michaille*. SIAM, Philadelphia, PA, 2014. $141.00. xii+793 pp., hardcover. ISBN 978-1-611973-47-1.

This is the second edition of a book first published in 2006. It is very impressive and contains much more than the title would suggest. The first version was already a very rich textbook in analysis, going from the basics (topologies and convergences, calculus of variations, measure theory, Sobolev spaces and capacities, convex analysis) to more involved topics ($BV$ and $SBV$ functions, lower semicontinuity and relaxation, Young measures, $\Gamma$-convergence), illustrated by the detailed analysis of a few simple variational problems. It also contained a complete chapter on the finite element method. The second edition has been substantially enriched with more examples, new theoretical tools such as an introduction to (variational) stochastic homogenization and to optimal transportation, and with a new chapter (of more than 100 pages!) on gradient flows.

Like its previous edition, this book is divided into two parts, titled "Basic Variational Principles" and "Advanced Variational Analysis." The first part contains nine chapters, starting with the fundamentals of analysis (distributions, topologies, lower-semicontinuity, including an original approach to Ekeland's variational principle), a selection of some well-presented measure theoretic tools (Carathéodory's construction and Hausdorff measures, Young measures, capacity theory), followed by the analysis of some variational PDEs and systems in various settings (Dirichlet, Neumann, or mixed boundary conditions and transmission conditions, nonlinearities, Lamé system, critical points, obstacles) with, in fact, many more examples than in the previous edition.

This part ends with a chapter on the basics of the finite element method and an introduction to standard error estimates (note that Galerkin's approximation has already been introduced in the third section, where it is used to give a simple proof of the Lax–Milgram theorem), a detailed chapter on the spectral decomposition of the Laplace operator, and a 55-page introduction to convex duality, in Chapter 9, which could be seen as a minitextbook in itself. Indeed, this last section is very complete and nicely introduces theoretical tools such as inf-convolutions, Legendre–Fenchel duality, subdifferential calculus, and practical optimization (multipliers, KKT conditions, dual problems, linear programs). It ends with a detailed introduction to convex-concave saddle points problems and Fenchel–Rockafellar duality.

The second part of the new edition now contains eight chapters. The first one, as before, is devoted to $BV$ and $SBV$ functions. Many important technical points, such as the rectifiability of the level sets, are proven, at least partially. $SBV$ functions are introduced to pave the way for the models of image segmentation and fracture growth which are introduced further on, in Chapter 14. Then follows a long and detailed chapter on lower semicontinuous relaxation, in particular, in $BV$ or measure spaces. A short theoretical introduction is followed by a section on integral functionals with $p$-growth, where important concepts such as Morrey's quasiconvexity are discussed (in fact, part of this discussion is spread between this chapter and a further one on lower semicontinuity); the Young measure approach to relaxation is further discussed in great detail in a quite substantial section. The case of relaxation for problems with growth 1, in the space $BV$ of functions with bounded variation, which requires more advanced measure-theoretical tools, is also briefly considered (and reappears two chapters later). An addition to the second edition is a very brief section on mass transportation, which introduces some essential notions and describes the Monge–Kantorovich duality. (This part might have been more appropriate as an illustration of convex duality in Chapter 9, or as a separate and more complete 18th chapter at the end of the book, where Wasserstein flows or Brenier's theorem could have been discussed.)

The next chapter introduces the notion of $\Gamma$-convergence (in metrizable spaces) and then quickly switches to useful applications such as 3D-2D limits or (variational) homogenization. Quite interesting in this new edition are the almost 30 pages on stochastic homogenization of minimization problems (with growth $p > 1$), which cover a topic rarely found in textbooks. This chapter ends with a brief description of Modica–Mortola and Ambrosio–Tortorelli approximations of perimeter/free discontinuity problems. Chapter 13 is devoted to the lower semicontinuity of integral functionals in the scalar and vectorial cases, and refines some of the results of Chapter 11 (maybe these parts could have been merged together, as there is some redundancy). It also addresses the issue of $SBV$ functions, which are used in some of the applications studied in the next chapter. Indeed, this next part, which is very interesting, shows how the previously introduced tools are used in practical examples such as (Hencky) plasticity, fracture mechanics, and the Mumford–Shah functional. Chapter 15 is quite original. It addresses the issue of coercivity and introduces tools for the study of noncoercive variational problems. In particular, it contains a very detailed analysis of the properties of recession functions of convex functions, which are not easily found elsewhere. Next comes an introduction to some shape optimization problems. A few interesting examples are given and the most useful technical tools to deal with a few fundamental optimization problems (with respect to a domain or a potential in an elliptic PDE) are described. Finally, Chapter 17 is entirely new and is an important addition to the book. It is devoted to gradient flows, mostly in the convex case, and contains fundamental notions as well as quite original material. Four important subjects are developed. It starts with classical results (Cauchy–Lipschitz theorem, asymptotics), followed by a quite complete description of convex gradient flows, with important tools from the theory of maximal monotone operators such as Moreau–Yosida approximation, Chernoff's lemma, a version of Opial's lemma, etc. This is illustrated by PDE examples such as the Stefan problem. A following section is devoted to the asymptotics of descent trajectories of real-analytic functions, based on the Kurdyka–Łojasiewicz inequality; the recent extension to the nonsmooth case (using semialgebraic functions) is also described. A third part studies limits of sequences of gradient flow problems, in the convex case (hence, based on Mosco-convergence of functionals), with an interesting application to stochastic homogenization in diffusion equations. Eventually, gradient flows in metric spaces are rapidly introduced together with the minimizing movement approach of De Giorgi, in a very short section which refers primarily to the well-known monograph of Ambrosio, Gigli, and Savaré.

All in all, this long textbook is very complete and pleasant to read, with a progressive level of difficulty and complexity and many nice examples which illustrate the theoretical results. It contains deep and precise information on many important tools in variational analysis (functional analysis, convex analysis) and many advanced methods, together with a general overview of most of the modern techniques. It should be useful for both students and researchers, whether they need to learn or review some advanced techniques in analysis or are looking for an introduction to more recent theories.

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