Preface

The topics of this book are election systems and fair division of resources. The book is not intended for readers who want a broad and sophisticated introduction to these large fields of mathematical economics, in which I am not by any means an expert. I have in mind a reader without any college-level mathematical background who will enjoy seeing examples of elementary mathematics applied to interesting nonmathematical questions. A few passages in small print are addressed to more advanced readers; see below. With the exception of those paragraphs, none of the material requires knowledge beyond elementary algebra, and most of it does not even require that. I do assume, however, that the reader is willing to make an effort to work through mathematical arguments which, while elementary, are not always easy.

The book grew out of notes for a course for liberal arts majors that I have taught at Tufts University several times. Mathematics has been applied to a vast array of important subjects, so why this particular choice of topics for a course aimed toward this group of students? My answer is that unlike most applications of mathematics, the topics in this book can be studied without extensive background knowledge. Nevertheless the reader will find some ideas and results that are less than fifty years old.

When I teach out of this text, I don't cover all chapters in a semester, but I do cover more than half. As laid out in detail below, many chapters are independent of each other, and Parts I, II, and III are independent of each other altogether. This makes it easy to be flexible, spend as much time as needed on a given chapter, give extra examples where needed, and compensate by omitting other chapters. It is my experience that liberal arts majors, even when they take the course for no reason other than to fulfill a mathematics distribution requirement, are entirely capable of absorbing and even enjoying many of the more theoretical topics and proofs, as long as the instructor goes slowly enough. The material is certainly a challenge for many of them; that is intentional.

In some places in the book, there are passages in small print. These require more knowledge, or more stamina in following mathematical reasoning, than I expect of students taking my course. They can be omitted without any loss of continuity, and they must definitely be omitted when teaching undergraduate liberal arts students with little mathematical background.

A small number of exercises are labeled with asterisks ($*$). These are exercises that I would consider too difficult to assign in my course for liberal arts majors.

I imagine that one could also teach a course for beginning undergraduate mathematics majors out of this text, although I have not yet tried that myself. One could then go faster, cover more chapters, put stronger emphasis on proofs, include some of the material in small print, and not shy away from assigning the exercises labeled with asterisks.

The chapters are grouped as follows.

Chapters 1–8: Elections for the purpose of selecting a single winner from a field of candidates. A fairly wide range of winner selection methods is introduced. A list of intuitively reasonable requirements narrows down the options, until we arrive at a single preferred method (preferred among those discussed in this book, not among all conceivable methods), Schulze's beatpath method. These chapters are best worked through in the order in which they are presented, without omissions.

Chapters 9 and 10: These chapters are among the hardest in the book, and can be omitted without losing continuity. They give examples of criteria that may seem attractive at first sight, but cannot be satisfied by any reasonable winner selection method. Theorem 10.3 is an easy result in this direction, and its statement and proof are easy to follow without reading anything else in Chapters 9 and 10, except for Definition 10.1.

Chapters 11 and 12: Elections for the purpose of ranking a field of candidates. Chapter 11 is a theoretical discussion of the relation between ranking and winner selection. Chapter 12 culminates in a proof of Arrow's famous dictatorship theorem.

Chapters 13–15 form the second part of the book. They are about "fair compensation," i.e., problems of the following kind. Two or more people jointly own some indivisible object, such as a house, a valuable painting, or a rabbit. They would like to transfer ownership to one person, who will have to compensate the others monetarily. One might think of a divorce or an inheritance. Who should receive ownership of the object, and what should the compensation payments be? This is obviously not a mathematical question, but one can formulate general principles that one might want to adhere to, and see, using mathematics, what follows from them. Chapter 13–15 preview, in this simple context, the notions of fairness, envy-freeness, Pareto-optimality, and equitability, which are fundamental to the following chapters on fair division. However, the same notions are introduced again, in the context of cake cutting, in Chapter 16, and therefore the third part of the book (Chapters 16–26) does not depend on the second.

Chapter 16 is an introduction to the "cake cutting" problem, that is, the problem of sharing a divisible resource among people with different tastes, preferences, or needs. The chapter introduces the fundamental notions of fairness, envy-freeness, Pareto-optimality, and equitability for cake divisions.

Chapters 17–20 describe generalizations of the "I cut, you choose" method of cake division. In the "I cut, you choose" method known to children in kindergarten, one child cuts the cake into two halves that she considers of equal value, and the other child chooses her half. The generalizations discussed in Chapters 17–20 have in common with "I cut, you choose" that the procedure starts with one person dividing the cake into pieces that she considers equal in value. A method of this sort for three people was proposed by Hugo Steinhaus in the 1940s (Chapter 17). Steinhaus's idea was extended to the case of arbitrarily many participants by Harold Kuhn in 1967 (Chapter 19). Chapter 18, on Hall's marriage theorem, is needed for Chapter 19. Chapter 20, which is independent of all other chapters in the book except for Chapter 16, describes a method for *envy-free* cake division among three people, invented by John Selfridge and John Conway.

In the rest of the book, Chapters 21–26, we assume that the cake consists of finitely many homogeneous pieces—some cheesecake and some chocolate cake, for instance. The important point about homogeneous pieces is that they are divisible *objectively*: We can disagree over whether chocolate mousse cake is more valuable than cheesecake or vice versa, but we cannot reasonably disagree over how much is one-quarter of eight ounces of chocolate mousse.

In Chapters 21–24, we describe and analyze the "adjusted winner method" of Steven Brams and Alan Taylor, quite arguably *the* right way for two people to divide a cake composed of finitely many homogeneous pieces. This method combines the three desirable properties of envy-freeness, Pareto-optimality, and equitability, and it is the only method with these three properties.

In Chapter 25, we study proportional allocation, an alternative way for two people to share a cake composed of finitely many homogeneous pieces. It is less easily manipulated by a dishonest person than the adjusted winner method, but it is not Pareto-optimal. It is best to read Chapters 21–24 prior to Chapter 25.

In Chapter 26, we study (necessarily imperfect) generalizations of the idea of the adjusted winner method to the case of more than two participants. It turns out that the three desirable properties of the adjusted winner method (envy-freeness, Pareto-optimality, and equitability) cannot all be guaranteed at the same time when there are more than two participants, but any two of the three can be. Chapter 26 answers questions that arise naturally from Chapters 21–23, but it is also among the most difficult chapters in the book, and can certainly be omitted.

There are appendices on set notation, logic, and mathematical induction. It is probably best to consult those when needed while reading the text, instead of studying them in advance. I also provide solutions to many of the homework exercises.

I am grateful to my wonderful colleague Martin Guterman, who created the course on Mathematics of Social Choice at Tufts. This book reflects my own way of teaching the course. While I don't know whether Marty, who passed away in 2004, would have liked it, I do know that he believed in teaching rigorous mathematics to students who major in fields far from mathematics, and I adopted this approach here. I am sure Marty would have been delighted to know that the course continues to draw large numbers of Tufts students every year.

I also owe gratitude to many students who have commented on my course notes and thereby helped improve them. I would like to mention one student by name, Sally Greenwald, who took the trouble of reading through the entire text one more time after the semester had ended (and she had already graduated) and gave me a long list of helpful comments.

Finally I would like to thank the excellent staff at SIAM's book program, especially Elizabeth Greenspan (senior acquisitions editor) and Lou Primus (production editor). They made the process of preparing the book for publication smooth and efficient. It was a great pleasure to work with them.

Christoph Börgers