# Preface

Hard constraints are represented by inequalities. Hard constraints occur frequently in practical systems and their models; these hard constraints are often used in optimization since optima are frequently found at these hard limits. In optimal control theory, this can be seen in the prevalence of "bang-bang" solutions—the "bang" represents a control at a hard limit.

In spite of this, hard limits are eschewed in most dynamical models. There are a number of reasons for this. One is the lack of a suitable or "nice" theory for such systems. Another is that numerical methods do not handle this situation well. A third is that it is often not clear what should happen in a differential equation when a hard limit is reached.

Most commonly, when researchers in the sciences, engineering, or economics come across a dynamic system with a hard constraint, the usual instinct is to smooth out the hard constraint, often by adding a "penalty" term for either violating, or approaching, the hard constraint. There are several reasons why this is not necessarily wise:

- This complicates an otherwise simple model, and furthermore, the strength of the penalty is a new parameter that should somehow be calibrated to fit the situation. And if the calibration indicates that the value should be zero (or infinity) for all practical purposes, then we are back to a system with a hard constraint.

- Numerically solving a penalized differential equation with a small penalty term (so as to well approximate a hard constraint) results in a stiff differential equation, which must be solved either with extremely small step sizes or using implicit methods that require the solution of nonlinear equations that are nearly equivalent to the hard constraint.

This book aims to fill this gap: hard limits are natural models for many dynamic phenomena, and there are ways of creating "differential equations with hard constraints" that are natural and provide accurate models of many physical, biological, and economic systems. The models that are described here have roots in optimization theory, and so we will see Lagrange multipliers and complementarity principles not only as methods to enable us to minimize functions subject to constraints but also as ways of formulating dynamic models to systems with hard constraints.

A central idea here is the idea of *index*. This represents the number of differentiations between a hard constraint and the state variables of the dynamic system. The higher the index, the more difficult it is to solve. This index is closely related to the index used in the area of *differential algebraic equations* (DAEs), which can be seen as differential equations with *equality constraints*. This will provide an organizing principle for much of this book.

Connections to related dynamical systems with hard constraints will be mentioned, such as linear complementarity systems (LCSs), projected dynamical systems (PDSs), differential inclusions, and more general concepts such as hybrid systems and variable structure systems. In contrast to hybrid and variable structure systems, the systems described here are more limited, but they do not suffer from the same theoretical limbo where solutions might or might not exist or be meaningful depending on the precise structure of the system.

Differential variational inequalities (DVIs) form a focal point in this book. These are not the most general class of dynamics that can represent hard constraints or impacts. Differential inclusions (also discussed) can be more general. But with such great generality it becomes difficult to turn the abstract formulation into a practical or computationally useful form. DVIs, on the other hand, provide a general means both of modeling and of carrying out computations. The connections with more abstract theories, such as differential inclusions, are fleshed out so that the reader can compare the properties and strengths of the two means of modeling such systems.

I have tried to make this book mathematically self-contained, so that it is accessible to engineers, economists, and others with a strong mathematical background. The material contained in this book, however, does involve considerable technical development, especially for problems involving partial differential equations, as these involve spaces of functions with infinite dimensions. Results are given which include infinite-dimensional cases wherever possible. If the reader does not have a background in functional analysis, regarding statements such as "… $X$ is a Banach space …," "… $X$ is a Hilbert space …," "… $X$ has the Radon–Nikodym property …," the reader should think of $\mathbb{R}^n$, the space of $n$-dimensional (column) vectors. The dual space $X'$ would then be the space of $n$-dimensional row vectors. The duality pairing between a vector $x \in X$ and $y^T \in X'$ is $\langle y^T, x\rangle = y^T x$, the usual inner product for $n$-dimensional vectors. The mysterious function $J_X$ (which takes vectors in a Hilbert space $X$ to the dual space $X'$) can be thought of as the transpose operation for $\mathbb{R}^n$. Weak and strong convergence are identical for finite-dimensional spaces, but they can be different in important ways in infinite-dimensional spaces. The relevant concepts and theorems are detailed in the appendices, along with relevant material on convex analysis and differential equations.

The results of mine here are often improvements of my published results. While I have aimed for generality, I have often not given the most general possible formulation. As this is a book, the target is readability and ease of understanding as well as completeness of the technical results.

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