Preface

This book intends to give an accessible account of applied mathematics, mainly of analysis subjects, with emphasis on functional analysis. The intended readers are senior or graduate students who wish to study analytical methods in science and engineering and researchers who are interested in functional analytic methods.

Needless to say, scientists and engineers can benefit from mathematics. This is not confined to a mere means of computational aid, and indeed the benefit can be greater and far-reaching if one becomes more familiar with advanced topics such as function spaces, operators, and generalized functions. This book aims at giving an accessible account of elementary real analysis, from normed spaces to Hilbert and Banach spaces, with some extended treatment of distribution theory, Fourier and Laplace analyses, and Hardy spaces, accompanied by some applications to linear systems and control theory. In short, it is a modernized version of what has been taught as applied analysis in science and engineering schools.

To this end, a more conceptual understanding is required. In fact, conceptual understanding is not only indispensable but also a great advantage even in manipulating computational tools. Unfortunately, it is not always accomplished, and indeed often left aside. Mathematics is often learned by many people as a collection of mere techniques and swallowed as very formal procedures.

This is deplorable, but from my own experience of teaching, its cure seems quite difficult. For students and novices, definitions are often difficult to understand, and mathematical structures are hard to penetrate, let alone the background motivation as to how and why they are formulated and studied.

This book has a dual purpose: one is to provide young students with an accessible account of a conceptual understanding of fundamental tools in applied mathematics. The other is to give those who already have some exposure to applied mathematics, but wish to acquire a more unified and streamlined comprehension of this subject, a deeper understanding through background motivations.

To accomplish this, I have attempted to

- elaborate upon the underlying motivation of the concepts that are being discussed and
- describe how one can get an idea for a proof and how one should formalize the proof.

I emphasized more verbal, often informal, explanations rather than streams of logically complete yet rather formal and detached arguments that are often difficult to follow for nonexperts.

The topics that are dealt with here are quite standard and include fundamental notions of vector spaces, normed, Banach, and Hilbert spaces, and the operators acting on them. They are in one way or another related to linearity, and understanding linear structures forms a core of the treatment of this book. I have tried to give a unified approach to real analysis, such as Fourier analysis and the Laplace transforms. To this end, distribution theory gives an optimal platform. The second half of this book thus starts with distribution theory. With a rigorous treatment of this theory, the reader will see that various fundamental results in Fourier and Laplace analyses can be understood in a unified way. Chapter 9 is devoted to a treatment of Hardy spaces. This is followed by a treatment of a remarkably successful application in modern control theory in Chapter 10.

Let us give a more detailed overview of the contents of the book. As an introduction, we start with some basics in vector spaces in Chapter 1. This chapter intends to give a conceptual overview and review of vector spaces. This topic is often, quite unfortunately, a hidden stumbling point for students lacking an in-depth understanding of what linearity is all about. I have tried to illuminate the conceptual sides of the notion of vector spaces in this chapter. Sometimes, I have attempted to show the idea of a proof first and then show that a complete proof is a "realization" of making such an idea logically more complete. I have also chosen to give detailed treatments of dual and quotient spaces in this chapter. They are often either very lightly touched on or neglected completely in standard courses in applied mathematics. I expect that the reader will become more accustomed to standard mathematical thinking as a result of this chapter.

From Chapter 2 on, we proceed to more advanced treatments of infinite-dimensional spaces. Among them, normed linear spaces are most fundamental and allow rather direct generalizations of finite-dimensional spaces. A new element here is the notion of norms, which introduces the concept of topology. Topology plays a central role in studying infinite-dimensional spaces and linear maps acting on them. Normed spaces give the first step toward such studies.

A problem is that limits cannot, generally, be taken freely for those sequences that may appear to converge (i.e., so-called Cauchy sequences). The crux of analysis lies in limiting processes, and to take full advantage of them, the space in question has to be "closed" under such operations. In other words, the space must be *complete*. Complete normed linear spaces are called Banach spaces, and there are many interesting and powerful theorems derived for them. If, further, the norm is derived from an inner product, the space is called a Hilbert space. Hilbert spaces possess a number of important properties due to the very nature of inner products—for example, the notion of orthogonality. The Riesz representation theorem for continuous linear functionals, as well as its outcome of the orthogonal projection theorem, is a typical consequence of an inner product structure and completeness. Hilbert space appears very frequently in measuring signals due to its affinity with such concepts as energy, and hence in many optimization problems in science and engineering applications. The problem of best approximation is naturally studied through the projection theorem in the framework of Hilbert space. This is also a topic of Chapter 3.

Discussing properties of spaces on their own will give only half the story. What is equally important is their interrelationship, and this exhibits itself through linear operators. In this connection, dual spaces play crucial roles in studying Banach and Hilbert spaces. We give in Chapter 5 a basic treatment of them and prove the spectral resolution theorem for compact self-adjoint operators—what is known as the Hilbert–Schmidt expansion theorem.

We turn our attention to Schwartz distributions in Chapter 6. This theory makes transparent the treatments of many problems in analysis such as differential equations, Fourier analysis (Chapter 7), Laplace transforms (Chapter 8), and Poisson integrals (Chapter 9), and it is highly valuable in many areas of applied mathematics, both technically and conceptually. In spite of this fact, this theory is often very informally treated in introductory books and thereby hardly appreciated by engineers. I have strived to explain why it is important and how some more rigorous treatments are necessary, attempting an easily accessible account for this theory while not sacrificing mathematical rigor too much.

The usefulness of distributions hinges largely on the notion of the delta function (distribution). This is the unity element with respect to convolution, and this is why it appears so frequently in many situations of applied mathematics. Many basic results in applied mathematics are indeed understood from this viewpoint. For example, a Fourier series or Poisson's integral formula is the convolution of an objective function with the Dirichlet or the Poisson kernel, and its convergence to such a target function is a result of the fact that the respective kernel converges to the delta distribution. We take this as a leading principle of the second half of this book, and I attempted to clarify the structure of this line of ideas in the treatments of such convergence results in Chapter 6, and subsequent Chapters 7 and 8 dealing with Fourier and Laplace transforms.

Chapter 9 gives a basic treatment of Hardy spaces, which in turn played a fundamental role in modern control theory. So-called H^∞ control theory is what we are concerned with. Of particular interest is generalized interpolation theory given here, which also plays a fundamental role in this new control theory. We will prove Nehari's theorem as well as Sarason's theorem, along with the applications to the Nevanlinna–Pick interpolation and the Carathéodory–Fejér theorem. We will also discuss the relationship with boundary values and the inner-outer factorization theorem. I have tried to give an easy entry point to this theory.

Chapter 10 is devoted to basic linear control system theory. Starting with an inverted pendulum example, we will see such basic concepts as linear system models, the concept of feedback, controllability, and observability, a realization problem, an input/output framework, and transfer functions, leading to the simplest case of H^∞ control theory. We will see solutions via the Nevanlinna–Pick interpolation, Nehari's theorem, and Sarason's theorem, applying the results of Chapter 9. Fourier analysis (Chapter 7) and the Laplace transforms (Chapter 8) also play key roles here. The reader will no doubt see part of a beautiful application of Hardy space theory to control and systems. It is hoped that this chapter can serve as a concise introduction to those who are not necessarily familiar with the subject.

It is always a difficult question how much preliminary knowledge one should assume and how self-contained the book should be. I have made the following assumptions:

- As prerequisite, I assumed that the reader has taken an elementary course in linear algebra and basic calculus. Roughly speaking, I assumed the reader is at the junior or a higher level in science and engineering schools.

- I did not assume much advanced knowledge beyond the level above. The theory of integration (Lebesgue integral) is desirable, but I chose not to rely on it.

- However, if applied very rigorously, the above principles can lead to a logical difficulty. For example, Fubini's theorem in Lebesgue integration theory, various theorems in general topology, etc., can be an obstacle for self-contained treatments.

I tried to give precise references in such cases and not overload the reader with such concepts.

- Some fundamental notions in sets and topology are explained in Appendix A. I tried to make the exposition as elementary and intuitive as to be beneficial to students who are not well versed in such notions. Some advanced background material, e.g., the Hahn–Banach theorem, is also given here.

This book is based on the Japanese version published by the Asakura Publishing Company, Ltd., in 1998. The present English version differs from the predecessor in many respects. Particularly, it now contains Chapter 10 for application to system and control theory, which was not present in the Japanese version. I have also made several additions, but it took much longer than expected to complete this version. Part of the reason lies in its dual purpose—to make the book accessible to those who first study the above subjects and simultaneously worthwhile for reference purposes on advanced topics. A good balance was not easy to find, but I hope that the reader finds the book helpful in both respects.

It is a pleasure to acknowledge the precious help I have received from many colleagues and friends, to whom I am so much indebted, not only during the course of the preparation of this book but also over the course of a long-range friendship from which I have learned so much. Particularly, Jan Willems read through the whole manuscript and gave extensive and constructive comments. I am also most grateful for his friendship and what I learned from numerous discussions with him. Likewise, I have greatly benefited from the comments and corrections made by Thanos Antoulas, Brian Anderson, Bruce Francis, Tryphon Georgiou, Nobuyuki Higashimori, Pramod Khargonekar, Hitay Özbay, Eduardo Sontag, and Mathukumalli Vidyasagar. I would also like to acknowledge the help of Masaaki Nagahara and Naoki Hayashi for polishing some proofs and also helping me in preparing some figures. I wish to acknowledge the great help I received from the editors at SIAM in publishing the present book.

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