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Featured Review: Linear and Nonlinear Functional Analysis with Applications. *By Philippe G. Ciarlet.* SIAM, Philadelphia, 2013. $98.00. xiv+832 pp., hardcover. ISBN 978-1-611972-58-0.

At the turn of the 20th century, the study of the classical problems of integral equations, ordinary and partial differential equations, and the calculus of variations, problems which frequently came from mathematical physics, was transformed. It came to be understood that many of these problems had a common core, the understanding of which would lead to a coherent theory capable of treating a much broader class of problems. In 1908, Maurice Fréchet named this body of knowledge "functional analysis." On the first page of his classic 1932 book [1], Stefan Banach tells us that in this theory the methods of classical mathematics are united with modern methods in a "perfectly harmonious and remarkably effective manner."

The goal of the book under review is to gather in a single volume the basic theorems of functional analysis, with detailed proofs and full background material, together with a wide range of their applications, with a particular focus on problems framed as partial differential equations or posed in variational form. Many, very different, books could be written on this range of topics. So, before putting pen to paper, the crucial and difficult choice had to be made of the specific material to be selected. Indeed, according to Peter Lax [5], Kurt Friedrichs remarked that it is very easy to write a book if you are willing to put into it everything you know about the subject. Wisely, Ciarlet did not write such a book. He chose a collection of some of the most important topics in functional analysis, together with selected applications well matched to the analysis, guided by his own research in several important areas. The book is carefully written with clarity and fluidity. Ciarlet has resisted the temptation to banish rather technical, but nonetheless important, details of proofs to the exercises. Between the mathematics are intervals in which he synthesizes the preceding material and foreshadows what will follow. The extensive bibliography runs to more than 25 pages: the references and corresponding commentary illuminate the genesis and development of several important areas of modern mathematics. There are more than 400 exercises, ranging from the routine to the challenging, many of which are guided extensions of material in the text.

The book will be valuable and interesting for researchers, particularly for those interested in numerical analysis, and for mathematically inclined scientists and engineers, since it clearly presents, in a single source and starting from first principles, the broad range of requisite background mathematics that is currently scattered among the existing literature. For a graduate course in purer functional analysis it provides a rich supply of examples, presented with proper rigor. For other courses—say, in elasticity, fluid mechanics, partial differential equations, numerical analysis, or computation—it provides students with a source for the requisite purer topics, presented in a manner congenial to their more particular interests.

One of the stimuli for the development of functional analysis[1] was an 1875 discovery of Weierstrass. Contrary to what he himself earlier believed impossible, a belief shared by Gauss, Kelvin, Dirichlet, and Riemann, Weierstrass exhibited a functional that was bounded below but failed to achieve a minimum value.[2]

In view of this example, what was to be made of the Dirichlet Principle? For a domain Ω in the plane, bounded by a smooth curve Γ, together with a continuous function $f\colon \Omega \to R$, determine a solution of

[1] For the history of functional analysis, see, say, Dieudonné [4], Monet [6], Pietsch [7], and Birkhoff [2].

[2] Weierstrass observed that the functional ψ that took the value $\psi(u) = \int_{-1}^{1} [xu'(x)]^2\, dx$ for differentiable functions u such that $u(-1) = -1, u(1) = 1$ certainly did not achieve the value 0. On the other hand, if $u_n(x) = \arctan(nx)[\arctan(n)]^{-1}$ for each n, then $\lim_{n\to\infty} \psi(u_n) = 0$.

Laplace's equation

$$\Delta u \equiv u_{x\,x} + u_{y\,y} = f \text{ in } \Omega,\ u = 0 \text{ on } \Gamma,$$

as a minimizer, among continuous functions u on $\Omega \cup \Gamma$ that equal 0 on Γ and have bounded continuous first derivatives in Ω, of the energy functional

$$(1) \quad \psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega f \cdot u\,dx.$$

This is the Dirichlet Principle, founded on the observation that if u_0 is a minimizer of the energy functional and the function ϕ belongs to $C_c^\infty(\Omega)$, the C^∞ functions on Ω that vanish in a neighborhood of Γ, then $\frac{d}{dt}\psi(u_0 + t\phi)|_{t=0} = 0$ so that

$$(2) \quad \int_\Omega \nabla u_0 \cdot \nabla \phi\,dx = \int_\Omega f \cdot \phi\,dx \quad \text{for all } \phi \text{ in } C_c^\infty(\Omega).$$

A function satisfying the above relationship is called a variational solution of Laplace's equation. This presentation of the Dirichlet Principle is certainly not convincing: it is not clear that the energy functional has a minimizer. Furthermore, even if it has a minimizer, it is not clear that this minimizer has the continuous second derivatives necessary to justify the integrations by parts that will directly show it is a pointwise solution of Laplace's equation.

In the late 19th century this discovery by Weierstrass stimulated work by some of the great mathematicians on partial differential equations and integral equations, including Laplace's equation. In the first decade of the 20th century, landmark papers on integral equations were written by Ivar Fredholm and by David Hilbert.

Fredholm considered, for a continuous function k on $[0, 1] \times [0, 1]$, the integral equation

$$u(x) + \int_0^1 k(x, y)u(y)\,dy = f(x) \quad \text{for } x \text{ in } [0, 1],$$

where f and u are continuous functions on $[0, 1]$. As but one of many results, he proved that there is a solution for every continuous f if and only if there is a unique solution in the case $f = 0$. He applied his integral operator theory to deduce existence and uniqueness results for a variety of partial differential equations, including Laplace's equation. Fredholm's results were based on a concept of determinant, a concept soon replaced by more general methods. Within twenty years the basic theorems for general continuous linear operators acting between Banach spaces—the Open Mapping Theorem, the Closed Graph Theorem, the Uniform Boundedness Principle, and the Hahn–Banach Theorem—were proved and appeared in Banach's 1932 book [1]. In Ciarlet's book these four theorems are among the theorems about continuous linear operators that are proved and used throughout, often in surprising niches, in the study of partial differential equations and variational problems. All of the necessary background in metric spaces, Banach spaces, Lebesgue integration, and the L^p spaces for $1 \le p \le \infty$ is presented in the initial chapters of the book.

Hilbert's approach to integral equations was quite different. Inspired by the Principle Axis Theorem for symmetric matrices and the progress that had been made in expressing quite general functions as series of eigenfunctions associated with Sturm–Liouville boundary value problems, he framed integral equations which had symmetric kernels in terms of continuous bilinear forms on the space ℓ^2 of square summable sequences, which are symmetric with respect to the natural inner-product that extends the Euclidean inner-product. In that context, he and his student Erhard Schmidt established a spectral theory for symmetric integral operators. They introduced duality into functional analysis, although it was somewhat hidden by the self-duality of ℓ^2, and also introduced the concept of weak convergence, proving the weak sequential compactness of the closed unit ball of ℓ^2. In another classic 1932 book [9], John von Neumann presented his theory of general Hilbert spaces and linear operators on these spaces, the centerpiece of which was his general spectral theorem for self-adjoint operators. In Ciarlet's book geometric properties of Hilbert spaces are considered, including orthogonal decompositions and orthogonal expansions, in general and concrete cases. As a basis for establishing minimization principles, the closest point property is proved: for every closed, convex subset of a Hilbert space and point outside the set, there is a unique point in the set which is closest to the outside point. The Riesz–Fréchet Representation

Theorem, which characterizes the dual of a Hilbert space, is proved. The Spectral Theorem for compact symmetric integral operators on ℓ^2 is extended to compact, symmetric operators on a general Hilbert space. This is used to study the eigenvalues of uniformly elliptic partial differential equations.

Very soon after the work of Hilbert and of Schmidt, Frigyes Riesz moved duality outside the ℓ^2 framework. He defined, for $1 \le p < \infty$, the $L^p[a, b]$ spaces with respect to Lebesgue measure, showed they were separable and complete, and, for $1 < p < \infty$, characterized their dual spaces and proved the weak sequential compactness of their closed balls. A few decades later, it became clear that generalizations of the L^p spaces preserving all the above properties, the Sobolev spaces, were the natural domains for important differential operators. In Ciarlet's book, properties of the Sobolev spaces, including trace spaces, the Embedding theorems, and compactness criteria, are presented without proof but are carefully explained. Dual spaces and weak convergence for sequences of functionals are examined, both in general and in concrete spaces. A complete proof, in the separable case, is presented of the characterization of reflexive Banach spaces as being those for which every bounded sequence has a weakly convergent subsequence.

Returning to the Dirichlet Principle, consider the functional $\psi\colon H_0^1(\Omega) \to \mathbf{R}$ defined by (1).[3] Then ψ is convex and continuous. The Poincaré–Friedrich inequality is established, from which it follows that ψ is bounded below and that minimizing sequences for this functional are bounded. Since $H_0^1(\Omega)$ is a Hilbert space, any bounded sequence has a subsequence that converges weakly. Moreover, as a consequence of the Hahn–Banach Theorem, for a sequence that converges weakly to u, there is a sequence of convex combinations of the terms of the sequence that converge strongly to u. Therefore, by the convexity and continuity of ψ, any minimizing sequence for ψ has a subsequence that converges weakly to a minimizer of ψ. This minimization strategy establishes a variational solution of Laplace's equation. It also establishes a variational solution of many other equations, including von Kármán's equation, a fourth-order partial differential equation occurring in continuum mechanics. The strategy and its variations, in particular, to reflexive Banach spaces, is presented in the book. The question of smoothness of minimizers of functionals is addressed: a proof is given of Weyl's Lemma which, as one consequence, tells us that variational solutions of Laplace's equation are smooth if f is smooth.

While the energy functional (1) is convex with a convex domain, in several applications it is necessary to consider functionals that are neither convex nor even defined on convex domains. In Ciarlet's book the minimization of an energy functional associated with certain problems in three-dimensional nonlinear elasticity is considered. For this problem both convexity of the domain and convexity of the functional are prohibited by the physics. In this particular case, the remarkable Polyconvexity Theorem of John Ball is proved. Ball discovered a property of this functional called polyconvexity that is compatible with the physics, and he proved the existence of a minimizer for it. The weak convergence of minimizing sequences to minimizers is established by verifying a delicate continuity property for certain nonlinear operators between carefully chosen Sobolev spaces, continuity with respect to the weak topology in the domain and the range. Details of all this are presented in the book.

Another approach to the analysis of the Dirichlet Principle is to use representations of linear operators to study the variational equation (2). Indeed, let H be a Hilbert space, $a\colon H \times H \to \mathbf{R}$ be a bilinear form, and $\ell\colon H \to \mathbf{R}$ be a linear functional. The following is called a quadratic minimization problem: find u_0 in H such that

(3)
$$\psi(u_0) = \inf_{u \in H} \psi(u), \text{ where}$$
$$\psi(u) = \frac{1}{2} a(u, u) - \ell(u) \quad \text{for all } u \text{ in } H.$$

If a is symmetric, the variational equation associated with (3) is: find u_0 in H such

[3] $H_0^1(\Omega)$ is the Hilbert space comprising the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_1$ defined by $\|u\|_1^2 = \int_\Omega u^2\,dx + \int_\Omega |\nabla u|^2\,dx$.

that

$$(4) \quad a(u_0, u) - \ell(u) = 0 \quad \text{for all } u \text{ in } H.$$

This is a generalization of the above framework for the study of Laplace's equation: set $H = H_0^1(\Omega)$, $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$ and $\ell(u) = \int_\Omega f \cdot u \, dx$. A bilinear form a is called coercive provided there is a $c > 0$ such that $a(u,u) \geq c\|u\|^2$ for all u in H. If a is bilinear, continuous, coercive, and symmetric, and ℓ is continuous, according to the Riesz Representation Theorem, there is a unique solution of (4). In this case, because of symmetry, (3) is equivalent to (4). In the absence of symmetry, the Lax–Milgram Lemma asserts that if a is bilinear, continuous, and coercive, and ℓ is continuous, there is a unique solution of (4). A variety of variational equations are considered in the book, among which are those posed for u belonging to a proper subset of H. Several boundary value problems are posed as variational equations, one example being the Stokes partial differential equations, which occur in the modeling of incompressible viscous fluids.

Of course, in general, there is not a useful variational formulation of the existence of solutions of nonlinear, or even linear, problems. For such problems, a natural first strategy is to extend concepts of calculus and topology to nonlinear operators between normed linear spaces. The Fréchet derivative is the natural extension of the derivative and such basic results as the Implicit Function Theorem, the companion Inverse Function Theorem, and Newton's Method extend to general nonlinear operators and are developed in the book. For mappings between metric spaces, classic iteration methods for solving differential equations are crystallized in the Banach Contraction-Mapping Theorem, and then there are topological techniques for the study of nonlinear problems. In the book the Brouwer Fixed Point Theorem is proved using purely analytical concepts, and it, together with a Galerkin argument, is then used to prove Schauder's extension of the Brouwer theorem: Every continuous map of a closed, compact subset K of a Banach space X into K has a fixed-point. Several applications of these calculus, metric, and topological results are provided.

The above is a précis of just a limited selection of topics in the book. As already observed, Ciarlet's book will be both useful and interesting for researchers and will also make an excellent text for graduate courses that emphasize the deep and fruitful relationship between functional analysis and its applications. A prominent ancestor is the venerable book of Frigyes Riesz and Béla Sz.-Nagy [8], and it is a welcome new companion of the more recent texts in the general area, among which are the books of Haim Brezis [3] and Peter Lax [5].

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