Homogeneous Coordinates

The parametrization

$$t \mapsto r(t) = \frac{\sum_{k=0}^{n} (c_k w_k)\, b_k^n(t)}{\sum_{k=0}^{n} w_k\, b_k^n(t)}, \quad 0 \le t \le 1\,,$$

of a rational Bézier curve can be identified with a polynomial parametrization

$$\tilde{r} = (p \,|\, q) = \sum_k (c_k w_k \,|\, w_k)\, b_k^n$$

in homogeneous coordinates, i.e., $r = (p_1, \ldots, p_d)/q$. This interpretation is convenient for implementing algorithms such as evaluation, differentiation, and subdivision. The procedures for polynomial Bézier curves are applied to $\tilde{r}$, and the result in $\mathbb{R}^{d+1}$ is projected to $\mathbb{R}^d$ by dividing by the last coordinate.

Differentation

The parametrization

$$r = \frac{\sum_{k=0}^{n} (w_k c_k)\, b_k^n}{\sum_{k=0}^{n} w_k\, b_k^n} = \frac{p}{q}$$

of a rational Bézier curve can be differentiated with the aid of Leibniz's rule:

$$\left(\frac{d}{dt}\right)^m (r(t)q(t)) = \sum_{\ell=0}^{m} \binom{m}{\ell} r^{(m-\ell)}(t)\, q^{(\ell)}(t) = p^{(m)}(t).$$

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This identity yields a recursion for $r^{(m)}$ in terms of the lower order derivatives:

$$\begin{aligned}
r' &= (p' - rq')/q, \\
r'' &= (p'' - 2r'q' - rq'')/q, \\
r''' &= (p''' - 3r''q' - 3r'q'' - rq''')/q, \\
\cdots & \quad .
\end{aligned}$$

For evaluating derivatives, we can therefore use the formulas and algorithms for standard Bézier curves. We simultaneously compute the derivatives of p and q and substitute the results into the recursion.