De Boor Algorithm

A linear combination of B-splines

$$p = \sum_k c_k \, b_k$$

of degree $\leq n$ with knot sequence ξ can be evaluated at $x \in [\xi_\ell, \xi_{\ell+1})$ by forming convex combinations of the coefficients of the B-splines which are nonzero at x. Starting with

$$p_k^0 = c_k, \quad k = \ell - n, \ldots, \ell\,,$$

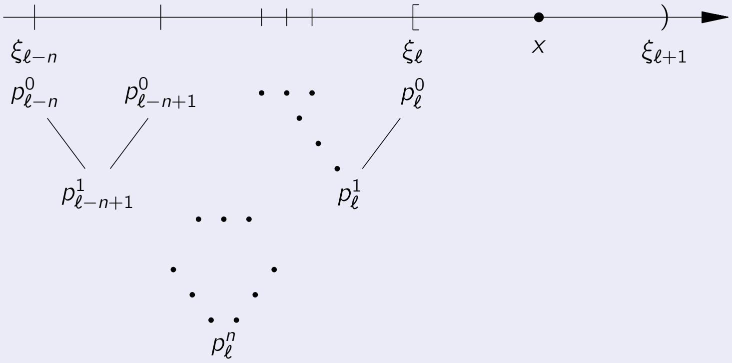
one computes successively, for $i = 0, \ldots, n-1$,

$$p_k^{i+1} = \gamma_{k,\xi}^{n-i} p_k^i + (1 - \gamma_{k,\xi}^{n-i}) p_{k-1}^i, \quad k = \ell - n + i + 1, \ldots, \ell\,,$$

with

$$\gamma_{k,\xi}^{n-i} = \frac{x - \xi_k}{\xi_{k+n-i} - \xi_k}$$

and obtains $p(x)$ as the final value p_ℓ^n.



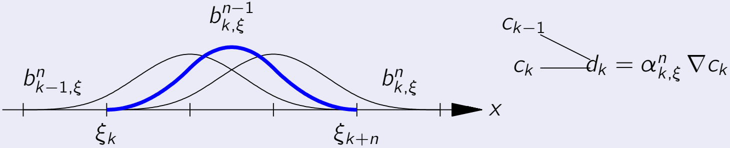
The triangular scheme is illustrated in the figure. It simplifies slightly if $x = \xi_\ell$. If ξ_ℓ has multiplicity μ, then $p(x) = p^{n-\mu}_{\ell-\mu}$; i.e., only $n - \mu$ steps of the recursion applied to $c_{\ell-n}, \ldots, c_{\ell-\mu}$ are needed.

Differentiation

The derivative of a spline is a linear combination of B-splines with the same knot sequence. More precisely, for x in any open knot interval $(\xi_\ell, \xi_{\ell+1})$ with $n+1$ relevant B-splines $b^n_{k,\xi}$,

$$\frac{d}{dx}\left(\sum_{k=\ell-n}^{\ell} c_k\, b^n_{k,\xi}(x)\right) = \sum_{k=\ell-n+1}^{\ell} \alpha^n_{k,\xi}\, \nabla c_k\, b^{n-1}_{k,\xi}(x)\,, \quad \alpha^n_{k,\xi} = \frac{n}{\xi_{k+n}-\xi_k}\,,$$

with ∇ the backward difference operator, i.e., $\nabla c_k = c_k - c_{k-1}$. The identity remains valid at the endpoints ξ_ℓ and $\xi_{\ell+1}$ of the knot interval if these knots have multiplicity $< n$.



For a spline $p = \sum_{k=0}^{m-1} c_k\, b_k \in S_\xi^n$ with knot sequence $\xi_0, \ldots, \xi_{m+n}$ and multiplicities $< n$,

$$p' = \sum_{k=1}^{m-1} d_k\, b_{k,\xi}^{n-1} \in S_{\xi'}^{n-1}\,,$$

where ξ' is obtained from ξ by deleting the first and last knots. This is consistent with the difference operation ∇, which reduces the range of indices by 1.