Schoenberg's Scheme

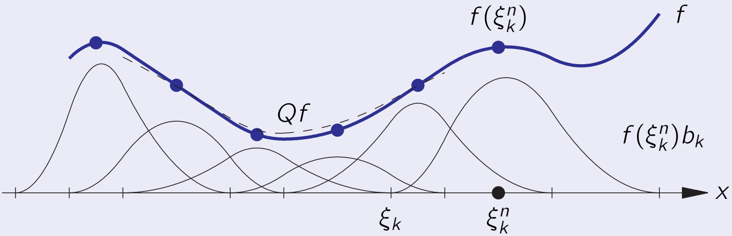
Schoenberg's scheme uses function values at the knot averages $\xi_k^n = (\xi_{k+1} + \cdots + \xi_{k+n})/n$ as coefficients of a spline approximation to a smooth function f:

$$f \mapsto Qf = \sum_{k=0}^{m-1} f(\xi_k^n)\, b_k \in S_\xi^n$$

with $\xi :\ \xi_0, \ldots, \xi_{m+n}\,.$
The method is second order accurate; i.e., for $x \in [\xi_\ell, \xi_{\ell+1}] \subseteq D_\xi^n = [\xi_n, \xi_m]$,

$$|f(x) - Qf(x)| \le \frac{1}{2}\, \|f''\|_{\infty, D_x}\, h(x)^2\,,$$

where $D_x = [\xi_{\ell-n}^n, \xi_\ell^n]$, $\|f''\|_{\infty, D_x}$ denotes the maximum norm of f'' on D_x, and $h(x) = \max(\xi_\ell^n - x, x - \xi_{\ell-n}^n)$.



The Schoenberg operator preserves positivity, monotonicity, and convexity. This means that

$$f^{(k)} \geq 0 \implies (Qf)^{(k)} \geq 0$$

for $k \leq 2$ if both derivatives are continuous. For a uniform knot sequence, the sign of all derivatives up to order n is preserved.