Natural Spline Interpolant

The natural spline interpolant of the data

$$(x_i, f_i), \quad x_0 < x_1 < \cdots < x_M,$$

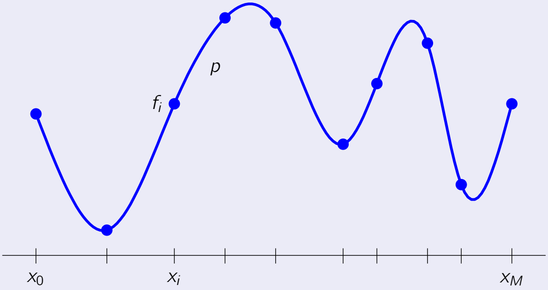
is a cubic spline p with simple knots at x_ℓ, which satisfies the boundary conditions

$$p''(x_0) = p''(x_M) = 0\,.$$

Among all twice continuously differentiable interpolants, p minimizes the integral

$$\int_{x_0}^{x_M} |p''(x)|^2\,dx\,,$$

which serves as a measure for the oscillations of p.



Alternatively, the boundary conditions

$$p'(x_0) = d_0, \quad p'(x_M) = d_M$$

are possible. The resulting clamped natural spline possesses the analogous extremal property.

Smoothing Spline

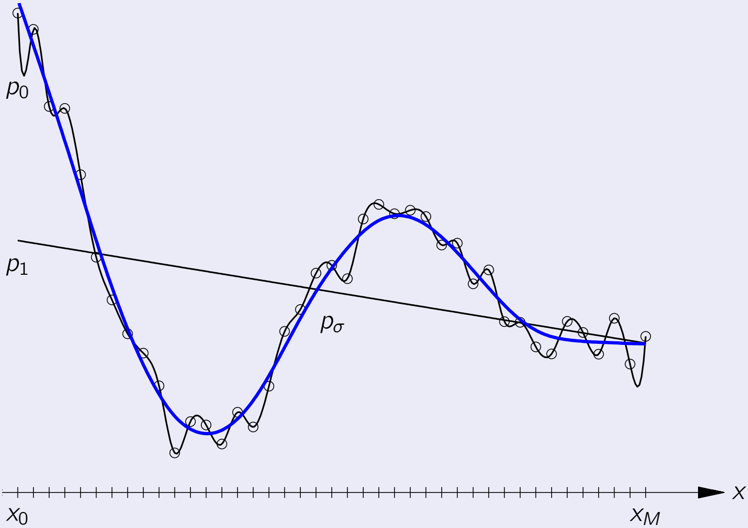
The smoothing spline p_σ for the data

$$(x_i, f_i), \quad x_0 < \cdots < x_M \,,$$

and the weights $w_i > 0$ is the unique cubic spline with simple knots at x_i which minimizes

$$E(p,\sigma) = (1-\sigma) \sum_{i=0}^{M} w_i \left| f_i - p(x_i) \right|^2 + \sigma \int_{x_0}^{x_M} \left| p'' \right|^2$$

among all twice continuously differentiable functions p.



The parameter $\sigma \in (0, 1)$ controls the significance of the data and of the smoothing. For $\sigma \to 0$, p_σ approaches the natural cubic spline interpolant, while, for $\sigma \to 1$, p_σ converges to the least squares line.