## Properties of the Control Polygon

The control polygon $c$ of a spline curve parametrized by

$$
p=\sum_{k=0}^{m-1} c_{k} b_{k}, \quad p_{\nu} \in S_{\tau}^{n}, \quad \tau: \tau_{0}, \ldots, \tau_{m+n}
$$

qualitatively models the shape of $p$.


As is illustrated by the figure,

- for $\tau_{\ell} \leq t \leq \tau_{\ell+1}$ the point $p(t)$ lies in the convex hull of $c_{\ell-n}, \ldots, c_{\ell}$. Moreover, if both endpoints of the parameter interval $D_{\tau}^{n}=\left[\tau_{n}, \tau_{m}\right]$ are knots with multiplicity $n$, then
- $p\left(\tau_{n}\right)=c_{0}, p\left(\tau_{m}\right)=c_{m-1}$,
- $p^{\prime}\left(\tau_{n}^{+}\right)=\alpha_{1, \tau}^{n}\left(c_{1}-c_{0}\right), p^{\prime}\left(\tau_{m}^{-}\right)=\alpha_{m-1, \tau}^{n}\left(c_{m-1}-c_{m-2}\right)$
with $\alpha_{k, \tau}^{n}=n /\left(\tau_{k+n}-\tau_{k}\right)$. The last two properties referred to as endpoint interpolation imply that the control polygon is tangent to the spline curve, which is very useful for design purposes.
While the parametrization of a spline curve is continuous, the derivative can have jumps. This is taken into account in the formula for $p^{\prime}$, where the superscripts $+/-$ denote limits from the right/left.


## Distance to the Control Polygon

For a spline curve parametrized by

$$
p=\sum_{k=0}^{m-1} c_{k} b_{k}, \quad p_{\nu} \in S_{\tau}^{n},
$$

with $n>1$, let $c$ be a piecewise linear parametrization of the control polygon, which interpolates $c_{k}$ at the knot averages $\tau_{k}^{n}=\left(\tau_{k+1}+\cdots+\tau_{k+n}\right) / n$.


The distance of $p$ to the control polygon can be bounded in terms of weighted second differences of the control points. For $t$ in a nondegenerate knot interval $\left[\tau_{\ell}, \tau_{\ell+1}\right] \subseteq D_{\tau}^{n}$,

$$
\|p(t)-c(t)\|_{\infty} \leq \frac{1}{2 n} \max _{\ell-n \leq k \leq \ell} \sigma_{k}^{2} \max _{\ell-n+2 \leq k \leq \ell}\left\|\nabla_{\tau}^{2} c_{k}\right\|_{\infty}
$$

where

$$
\sigma_{k}^{2}=\frac{1}{n-1} \sum_{i=1}^{n}\left(\tau_{k+i}-\tau_{k}^{n}\right)^{2}
$$

and $\nabla_{\tau}^{2} c_{k}$ are the control points of the second derivative $p^{\prime \prime}$.
Their explicit form is

$$
\nabla_{\tau}^{2} c_{k}=\frac{n-1}{\tau_{k+n-1}-\tau_{k}}\left(\frac{c_{k}-c_{k-1}}{\tau_{k}^{n}-\tau_{k-1}^{n}}-\frac{c_{k-1}-c_{k-2}}{\tau_{k-1}^{n}-\tau_{k-2}^{n}}\right)
$$

where none of the denominators vanishes since the differences are at least as large as $\tau_{\ell+1}-\tau_{\ell}$.

The local estimate implies a global bound simply by taking the maximum of the right side over all $k$ relevant for the entire parameter interval $D_{\tau}^{n}=\left[\tau_{n}, \tau_{m}\right]$ of the spline curve. In this case, $\nabla_{\tau}^{2} c_{k}$ is set to 0 if $\tau_{k}=\tau_{k+n-1}$.

