Knot Insertion

Let $p = \sum_0^{m-1} c_k\, b_k \in S_\tau^n$ parametrize a spline curve. If we add a new knot s in the parameter interval D_τ^n and $s \in [\tau_\ell, \tau_{\ell+1})$, then the control points $\tilde{c}_k$ of p with respect to the refined knot vector

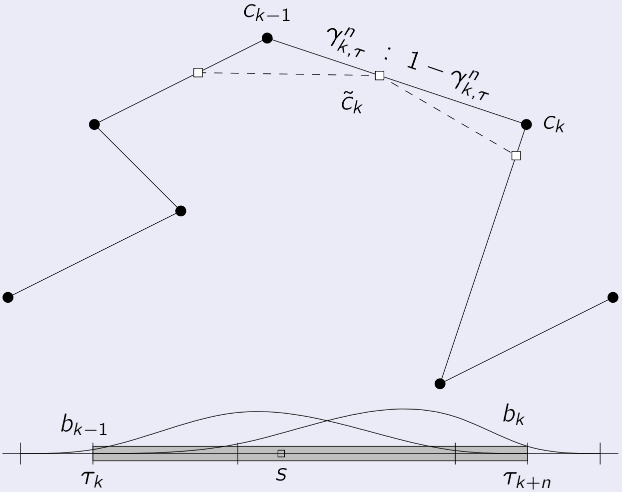
$$\tilde{\tau}: \quad \ldots,\, \tilde{\tau}_\ell = \tau_\ell,\, \tilde{\tau}_{\ell+1} = s,\, \tilde{\tau}_{\ell+2} = \tau_{\ell+1},\, \ldots$$

are computed as follows.
On the segments $[c_{k-1}, c_k]$ with $\tau_k < s < \tau_{k+n}$ new control points are generated:

$$\tilde{c}_k = \gamma_{k,\tau}^n c_k + (1 - \gamma_{k,\tau}^n) c_{k-1}, \quad \gamma_{k,\tau}^n = \frac{s - \tau_k}{\tau_{k+n} - \tau_k}\,.$$

The other edges of the control polygon remain unchanged:

$$\tilde{c}_k = c_k \text{ for } \tau_{k+n} \le s, \quad \tilde{c}_k = c_{k-1} \text{ for } s \le \tau_k\,.$$



As illustrated in the figure, the new control point $\tilde{c}_k$ divides the segment $[c_{k-1}, c_k]$ in the same ratio as the parameter s divides the interval $[\tau_k, \tau_{k+n}]$, which is the intersection of the supports of the two associated B-splines.

We note that, if s coincides with a knot ($s = \tau_\ell$), fewer control points need to be computed. More precisely, if s has multiplicity j in $\tilde{\tau}$, only $n + 1 - j$ convex combinations need to be formed.
Several new knots can be inserted by repeating the procedure. In particular, by raising the multiplicity of a knot to n, we obtain a point on the curve:

$$\tau_{\ell-n} < \tau_{\ell-n+1} = \cdots = \tau_\ell < \tau_{\ell+1} \implies p(\tau_\ell) = c_{\ell-n} .$$

Hence, the evaluation scheme for splines can be viewed as n-fold knot insertion.

Uniform Subdivision

Let

$$p = \sum_k c_k \, b_{k,\tau}^n \in S_\tau^n$$

be a parametrization of a uniform spline curve with grid width h. If new knots are simultaneously inserted at the midpoints of the knot intervals, the control points $\tilde{c}_k$ corresponding to the refined knot sequence $\tilde{\tau}$ with grid width $h/2$ and $\tilde{\tau}_{2k} = \tau_k$ can be computed as follows.

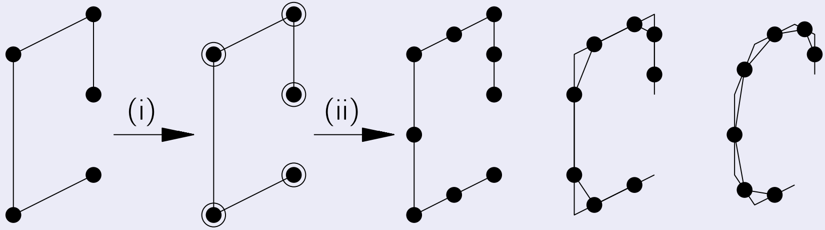
(i) The control points c_k, $k \sim \tau$, are doubled:

$$\tilde{c}_{2k} = \tilde{c}_{2k+1} = c_k \, .$$

(ii) Simultaneous averages of adjacent control points are formed,

$$\tilde{c}_k \leftarrow (\tilde{c}_k + \tilde{c}_{k-1})/2 \, ,$$

and this process is repeated n-times.



The explicit form of the new control points is

$$\tilde{c}_k = \sum_i s_{k-2i}\, c_i, \quad s_j = 2^{-n} \binom{n+1}{j},$$

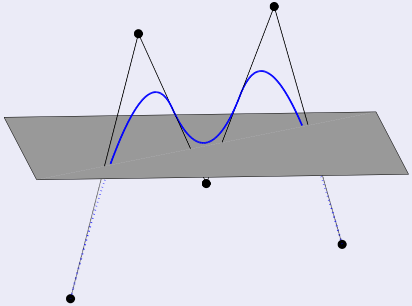
where $s_j = 0$ for $j < 0$ or $j > n+1$ in accordance with the convention for binomial coefficients.

Variation Diminution

The variation of a spline curve parametrized by $p = \sum_{k=0}^{m-1} c_k\, b_k$ with respect to a hyperplane H is not larger than the variation of its control polygon c:

$$V(p, H) \leq V(c, H)\,,$$

where V denotes the maximal number of pairs of consecutive points on opposite sides of H.



In particular, if the entire control polygon lies on one side of H, so does the spline curve.