Evaluation and Differentiation

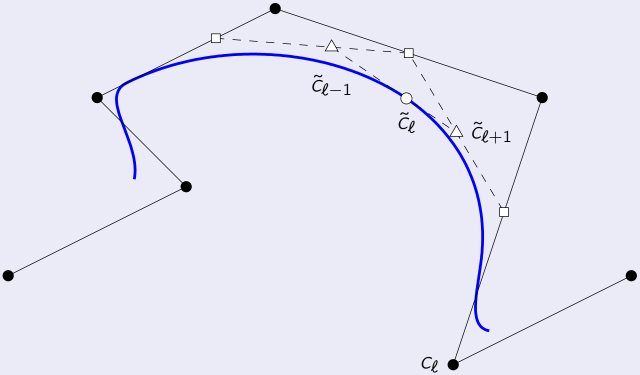
A point

$$p(s) = \sum_{k=0}^{m-1} c_k \, b_k(s)$$

on a spline curve with knot sequence $\tau : \tau_0, \ldots, \tau_{m+n}$ can be computed by repeatedly inserting s as a new knot until its multiplicity becomes n:

$$\tilde{\tau}_\ell < \tilde{\tau}_{\ell+1} = \cdots = \tilde{\tau}_{\ell+n} = s < \tilde{\tau}_{\ell+n+1} \implies p(s) = \tilde{c}_\ell \,,$$

where $\tilde{\tau}_\ell$ and $\tilde{c}_k$ denote the modified knots and control points, respectively.


$\tilde{c}_{\ell-1}$
$\tilde{c}_\ell$
$\tilde{c}_{\ell+1}$
c_ℓ

The refined control polygon $\tilde{c}$ is tangent to the curve:

$$p'(s^-) = \frac{n(\tilde{c}_\ell - \tilde{c}_{\ell-1})}{s - \tilde{\tau}_\ell}, \quad p'(s^+) = \frac{n(\tilde{c}_{\ell+1} - \tilde{c}_\ell)}{\tilde{\tau}_{\ell+n+1} - s},$$

where the one-sided derivatives coincide if s is not a knot with multiplicity n of the original knot sequence τ (i.e., if at least one knot is inserted). In this case,

$$p'(s) = \frac{n}{\tilde{\tau}_{\ell+n+1} - \tilde{\tau}_\ell} \left(\tilde{c}_{\ell+1} - \tilde{c}_{\ell-1}\right)$$

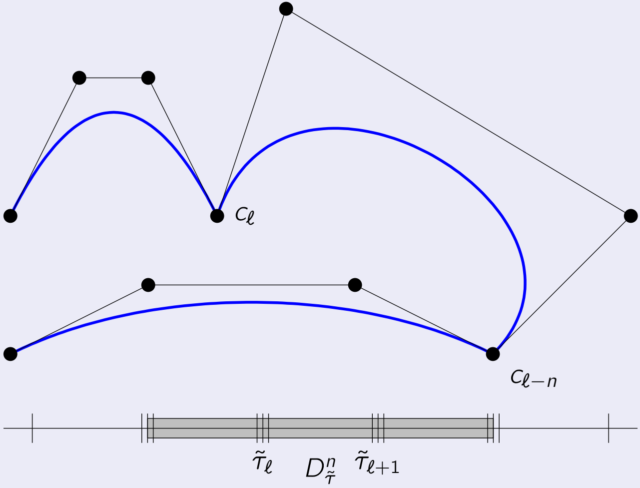
is an alternative formula for the tangent vector.

Bézier Form

The Bézier form of a spline curve parametrized by $p = \sum_{k=0}^{m-1} c_k\, b_k \in S_\tau^n$ is obtained by raising the multiplicity of each knot τ_k in the parameter interval $D_\tau^n = [\tau_n, \tau_m]$ to n. Then, for t in a nondegenerate parameter interval $[\tilde{\tau}_\ell, \tilde{\tau}_{\ell+1}] \subseteq D_\tau^n$ of the refined knot sequence $\tilde{\tau}$,

$$p(t) = \sum_{k=0}^{n} \tilde{c}_{\ell-n+k}\, b_k^n(s), \quad s = \frac{t - \tilde{\tau}_\ell}{\tilde{\tau}_{\ell+1} - \tilde{\tau}_\ell} \in [0, 1]\,,$$

where b_k^n are the Bernstein polynomials and $\tilde{c}_k$ the control points with respect to $\tilde{\tau}$. Hence, up to linear reparametrization (which is immaterial for the shape of the curve), the spline segments have Bézier form.



As shown in the figure, every nth control point lies on the curve separating the Bézier segments. Thus, by converting to Bézier form, we can apply polynomial algorithms simultaneously on the different knot intervals.