Multivariate Polynomials

A (real) polynomial p of d variables and coordinate degree $n = (n_1, \ldots, n_d)$ is a linear combination of monomials,

$$p(x) = \sum_{k \le n} c_k \, x^k, \quad x^k = x_1^{k_1} \cdots x_d^{k_d} \, ,$$

with coefficients $c_k \in \mathbb{R}$ and $c_n \neq 0$. The sum is taken over all multi-indices k (nonnegative integer vectors) with $k_\nu \le n_\nu$; i.e., p is a univariate polynomial of degree n_ν in each coordinate direction.

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The d-variate polynomials of coordinate degree $\leq n$ form a linear vector space, denoted by $\mathbb{P}^n$, of dimension $(n_1 + 1) \cdots (n_d + 1)$. More precisely, we write $\mathbb{P}^n(D)$ if the variable x is restricted to a particular domain $D \subseteq \mathbb{R}^d$.

Multivariate Bernstein Polynomials

The d-variate Bernstein polynomials of coordinate degree $(n_1, \ldots, n_d)$ are products of univariate Bernstein polynomials:

$$b_k^n(x) = \prod_{\nu=1}^{d} b_{k_\nu}^{n_\nu}(x_\nu), \quad 0 \le k_\nu \le n_\nu\,,$$

for x in the standard parameter domain $D = [0,1]^d$. They form a basis for $\mathbb{P}^n(D)$, which is symmetric with respect to the domain boundaries.

The properties of univariate Bernstein polynomials extend to d variables. In particular,

$$b_k^n \geq 0, \quad \sum_k b_k^n = 1\,.$$

Moreover, at a vertex $\sigma \in \{0,1\}^d$ of D,

$$b^n_{(\sigma_1 n_1, \ldots, \sigma_d n_d)}(\sigma) = 1\,,$$

and all other Bernstein polynomials vanish at σ.