Approximation

Univariate spline approximation schemes,

$$f \mapsto p = \sum_{k=0}^{m-1} c_k \, b_k \,,$$

which can be described by a matrix operation $c = Af$, can be combined to construct approximations with multivariate splines

$$p = \sum_{k_1=0}^{m_1-1} \cdots \sum_{k_d=0}^{m_d-1} c_k \, b_{k,\xi}^{n} \in S_{\xi}^{n} \,,$$

where $\xi_\nu : \xi_{\nu,0}, \ldots, \xi_{\nu,m_\nu+n_\nu}$.

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where $\xi_\nu : \xi_{\nu,0}, \ldots, \xi_{\nu,m_\nu+n_\nu}$. For a d-dimensional array of data f, the multivariate scheme $f \mapsto p$ is defined by

$$c_{(k_1,\ldots,k_d)} = \sum_{\ell_1} \cdots \sum_{\ell_d} a_{k_1,\ell_1}^1 \cdots a_{k_d,\ell_d}^d \, f_{(\ell_1,\ldots,\ell_d)} \, ,$$

where $A^1, \ldots, A^d$ are the matrices describing the univariate scheme, based on the knot sequences ξ_ν and the degrees n_ν.

This amounts to an application of the univariate scheme in each of the components with the indices corresponding to the other components held fixed. More precisely, the B-spline coefficients are computed in d steps:

$$f = f^0 \to f^1 \to \cdots \to f^d = c\,,$$

where

$$f^\nu(\ldots, k_\nu, \ldots) = \sum_{\ell_\nu} a^\nu_{k_\nu,\ell_\nu}\, f^{\nu-1}(\ldots, \ell_\nu, \ldots)$$

for $\nu = 1, \ldots, d$.

Error of Multivariate Spline Approximation

Assume that the local mesh ratio of the knot sequences ξ_ν of a spline space S_ξ^n is bounded by r. Then, any smooth function f can be approximated on the parameter hyperrectangle $R = D_\xi^n$ with optimal order:

$$|f(x) - p(x)| \leq c(n, r, d) \sum_{\nu=1}^{d} h_\nu(x)^{n_\nu+1} \, \|\partial_\nu^{n_\nu+1} f\|_{\infty,R} \quad \forall x \in R,$$

for some multivariate spline $p \in S_\xi^n$ and with $h_\nu(x)$ the width of the grid cell containing x in the νth coordinate direction.