APPROXIMATION AND MODELING 
WITH B-SPLINES 

Problem Collection 

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Part I

Problems
1 Polynomials

1.1 Monomial Form

Problem 1.1.1 Nested Multiplication for Hypergeometric Sums
Describe an algorithm for computing
\[ \sum_{k=0}^{n} \binom{a}{k} b_k x^k \]
with \((t)_\ell = t(t+1) \cdots (t+\ell-1)\) the Pochhammer symbol. How many operations are needed?

Answer
operations: \(n\)

1.2 Taylor Approximation

Problem 1.2.1 Error of the Taylor Polynomial for the Logarithm
Determine an upper bound for the error of the Taylor polynomial of degree \(\leq n\) at \(x_0 = 10\) for the natural logarithm on the interval \([10,11]\).

Answer
bound for \(n = 4\):

Problem 1.2.2 Table of the Exponential via Taylor Approximation
How small should \(h\) be chosen in order to approximate \(\exp(x), 0 \leq x \leq 1\), from tabulated values at the points \(h/2, 3h/2, 5h/2, \ldots, 1 - h/2\) by quadratic Taylor polynomials with error \(\leq 10^{-12}\)?

Answer
\(h \leq \) (based on the estimate for the Taylor remainder)

1.3 Interpolation

Problem 1.3.1 Quadratic Interpolation of Inaccurate Data
Estimate \(f(0)\) by interpolating the data

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

with a quadratic polynomial \(p\). In the worst case, by how much can \(p(0)\) differ if an inaccurate value of \(f(3)\) with relative error \(\leq 5\%\) is interpolated?

Answer
maximal difference:

Problem 1.3.2 4-Point Scheme
If the 4-point scheme is applied to nonperiodic data, for the first and last midpoints different formulas are needed. Determine the weights for the appropriate approximation

\[ f_{1/2} \approx \alpha f_0 + \beta f_1 + \gamma f_2 + \delta f_3. \]

Answer
maximum of the weights \(\alpha, \beta, \gamma, \delta\):
1.3.3 Program: Monomial Form of an Interpolating Polynomial

Write a program which generates the coefficients of an interpolating polynomial via the Aitken–Neville scheme.

Answer

smallest coefficient of the interpolating polynomial to \((k, \exp(k)), \ k = 0, \ldots, 4:\)

1.4 Bernstein Polynomials

1.4.1 Conversion of a Quartic Polynomial to Bernstein Form

Write the polynomial

\[ x \mapsto 3x^2 - 2x^3 - x^4 \]

as linear combination of quartic Bernstein polynomials.

Answer

sum of the Bernstein coefficients:

1.4.2 Scalar Product of Bernstein Polynomials

Derive a formula for the scalar product

\[ \int_0^1 b_k^n(x)b_j^m(x) \, dx \]

of two Bernstein polynomials.

Answer

\[ \int_0^1 b_3^3b_2^1: \]

1.5 Properties of Bernstein Polynomials

1.5.1 Integrals of Bernstein Polynomials

Compute

\[ a) \int_0^1 4b_1^3(t) - 5b_2^3(t) \, dt, \quad b) \int_0^x b_2^3(t) \, dt. \]

Answer

a) \[ \] b) value for \(x = 1:\)

1.5.2 Positivity of a Polynomial in Bernstein Form

Show by example that the positivity of a polynomial on the interval \([0, 1]\) does not imply the positivity of its Bernstein coefficients.
1.6 Hermite Interpolant

Problem 1.6.1 Hermite Representation and Volume of a Wine Bottle
Represent the contour of the wine bottle, depicted in the figure, as a Hermite spline and compute its volume.

![Wine Bottle Diagram]

Answer
Volume: \[\text{Volume} = \text{Area} \times \text{Height}\]

Problem 1.6.2 Approximation of a Square Root from Hermite Data
Determine an approximation to $\sqrt{x}$ for $x = 2$ from Hermite data at $x = 1, 4$.

Answer
Approximation for $\sqrt{2}$: \[\approx \text{value}\]

1.7 Approximation of Continuous Functions

Problem 1.7.1 Accuracy of the Bernstein Approximation
Determine the accuracy of the Bernstein approximation

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} f(k/n) b_n^k(x)$$

numerically. To this end, plot for the functions

- $a) f(x) = |2x - 1|$,
- $b) f(x) = \sqrt{x}$,
- $c) f(x) = \exp(x)$

the error $e_n = \max_{x \in [0,1]} |f(x) - p_n(x)|$ for $n = 1, \ldots, 8$.

Answer
Smallest of the errors, computed at $x = 0, 0.01, 0.02, \ldots, 1$: \[\text{value}\]

Problem 1.7.2 Convergence of Derivatives of the Bernstein Approximation
Show that, for a smooth function $f$, the derivatives of the Bernstein approximations

$$p_n = \sum_{k=0}^{n} f(k/n) b_n^k$$

converge to $f'$. 
2 Bézier Curves

2.1 Control Polygon

Problem 2.1.1 Vertex and Symmetry Axis of a Parabola
Determine the vertex and the symmetry axis of the parabola parametrized by
\[ p = (1, 0) b_1^2 + (0, 2) b_2^2. \]

Answer
axis of symmetry: \( \parallel (0, 1) \)

Problem 2.1.2 Nonexistence of Polynomial Parametrizations for a Circle
Show that a circular arc cannot be represented as a (polynomial) Bézier curve.

2.2 Properties of Bézier Curves

Problem 2.2.1 Tangent for a Bézier Curve with Multiple Control Points
Show that, for a Bézier curve with a multiple control point
\[ c_0 = c_1 = \cdots = c_{k-1} \neq c_k, \]
the tangent at the left endpoint is parallel to \( c_k - c_0 \). Does the curve, in general, possess a smooth regular parametrization?

Answer
number of continuous derivatives of a regular parametrization \( q (|q'(0)| \neq 0) \):

Problem 2.2.2 Cubic Bézier Approximation of a Semi-Circle
Determine the control points of a cubic Bézier curve \( p \) which touches a semi-circle at the points
\[ (-1, 0), (0, 1), (1, 0), \]
and compute the maximal deviation of \(|p(t)|\) from the radius 1.

Answer
\[ \max_{0 \leq t \leq 1} |p(t)| - 1: \]

2.3 Algorithm of de Casteljau

Problem 2.3.1 Evaluation of a Cubic Bézier Curve with de Casteljau’s Algorithm
Evaluate the cubic Bézier curve with control points
\[
\begin{pmatrix}
0 & 1 & -1 & 0 \\
0 & 1 & 1 & 0
\end{pmatrix}^t
\]
at \( t = 1/3 \) with de Casteljau’s algorithm.

Answer
smallest coordinate:
2.4 Differentiation

Problem 2.4.1 Approximation of the Exponential Function by a Quadratic Bézier Curve
Approximate the graph of the exponential function \( C : y = \exp(x) \) by a regular quadratic Bézier curve \( t \mapsto (x(t), y(t)) \) near \( x = 0 \), i.e., determine the Taylor coefficients of \( x \) and \( y \) so that
\[
y(t) - \exp(x(t)) = O(t^m)
\]
with \( m \) as large as possible. Compare with the accuracy of the quadratic Taylor polynomial \( x \mapsto y = 1 + x + x^2/2 \).
\[\text{Answer}\]
maximal order \( m: \boxed{} \)

Problem 2.4.2 Program: Distance from a Bézier Curve
Write a program which computes the distance of a point from a Bézier curve.
\[\text{Answer}\]
distance of \((1, 1)\) from the Bézier curve with control points \((0, 1), (0, 0), (2, 0)\): \boxed{}

2.5 Curvature

Problem 2.5.1 Curvature at the Midpoint of a Cubic Bézier Curve
Compute the curvature of the Bézier curve with control points
\[
C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]
at the midpoint of the parameter interval.
\[\text{Answer}\]
\(\kappa(1/2): \boxed{} \)

Problem 2.5.2 Smooth Extension of a Quadratic Bézier Curve
For the control points
\[
C^- = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\]
determine all control points \( C^+ \) for which the corresponding quadratic Bézier curves \( p^\pm \) join with second order contact at \((0, 0)\).
\[\text{Answer}\]
\(c_1^+ = (2, 0) \leadsto c_2^+ = (?, \boxed{}) \)

Problem 2.5.3 Bézier Curve with Second Order Contact with a Circle
Determine all regular planar quadratic Bézier curves which have second order contact with the unit circle at their endpoint \( p(0) = (1, 0) \).
\[\text{Answer}\]
\(c_1 = (?, 2) \leadsto c_2 = (\boxed{}, ?) \)
2.6 Subdivision

Problem 2.6.1 Program: Extension of Bézier Curves via an Inversion of de Casteljau’s Algorithm

Explain how to invert de Casteljau’s subdivision scheme

\[ p \mapsto (p_{\text{left}}, p_{\text{right}}), \]

i.e., how to determine \( p \) from \( p_{\text{left}} \). Write a program which computes the control points \( d_k \) of the extension of the Bézier curve with control points \( c_k \) to the interval \([0, s]\), \( s > 1 \).

**Answer**

largest coordinate of \( d_k \) for \( s = 4 \) and \( C = [0\, 2; 1\, 5; 3\, 6] \):

Problem 2.6.2 Reduction of the Edge Length of the Control Polygon by de Casteljau’s Algorithm

Show that the maximal edge length of the control polygon of a Bézier curve is reduced by at least a factor 1/2 by subdivision at the midpoint.

2.7 Geometric Hermite Interpolation

Problem 2.7.1 Geometric Hermite Interpolation of an Ellipse

Approximate the right half of the ellipse

\[ E : x_1^2 + x_2^2/4 = 1 \]

by a cubic Bézier curve via geometric Hermite interpolation.

**Answer**

first coordinate of the middle control points \( c_1 \) and \( c_2 \):
3 Rational Bézier Curves

3.1 Control Polygon and Weights

Problem 3.1.1 Interpolation with a Rational Quadratic Bézier Curve
Show that, with an appropriate choice of the weights, a rational quadratic Bézier curve can interpolate any point in the interior of the triangle formed by its control points. Determine, in particular, weights for the curve which passes through the center \((c_0 + c_1 + c_2)/3\) of the control polygon.

Answer
middle weight for a parametrization in standard form:

Problem 3.1.2 Limit for Movement of Weight Points
Determine the limit of a point \(r(t), \ t \in (0,1)\), on a rational Bézier curve as a weight point \(d_k\) is moved towards the control point

a) \(c_{k-1}\),  b) \(c_k\)

and the position of the other weight points is kept fixed.
Consider the example

\[
C = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
1 & 2 & 3 & 2 & 1
\end{pmatrix}^t
\]

and \(k = 3, \ t = 2/3\).

Answer
largest coordinate of the limit point for the example: a) , b) 

Problem 3.1.3 Determination of Curve Points via Affine Invariance
Determine the points \(r(1/3)\) for the three rational Bézier curves \(r\) with control and weight points shown in the figure.

Answer
largest coordinate of the three points: 

3.2 Basic Properties

Problem 3.2.1 Rational Cubic Parametrizations of a Semi-Circle
Determine a rational cubic parametrization of a semi-circle with weights in standard form.

Answer
middle weights:
Problem 3.2.2 Limit of a Rational Bézier Curve as Two Weights Approach Infinity
Determine the limit of a point \( r(t) \) of a rational Bézier curve as the weights of two inner control points tend to infinity with the same rate \( (w_j = w_k = \lambda \to \infty) \) and the other weights remain fixed. Consider
\[
C = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix}, \quad j = 1, \ k = 2
\]
and \( t = 1/3 \) as a concrete example.

**Answer**
largest coordinate of the limit point:

Problem 3.2.3 Rational Parametrization of a Polynomial Bézier Curve
Show that a rational Bézier parametrization with weights \( \gamma^k, \gamma \in (0, \infty) \), describes a polynomial curve.

3.3 Algorithms

Problem 3.3.1 Subdivision of a Rational Bézier Curve
Subdivide the rational Bézier curve with
\[
(C \mid w) = \begin{pmatrix} 5 & 7 & 3 \\ 0 & 2 & 9 \\ 8 & 10 & 6 \end{pmatrix}
\]
at the parameter \( t = 1/3 \).

**Answer**
largest coordinate of the common control point:

Problem 3.3.2 Program: Offsets of a Rational Bézier Curve
Write a program which plots the offset curves (parallel curves with prescribed distance) of a planar rational Bézier curve ignoring self-intersections.

3.4 Conic Sections

Problem 3.4.1 Distinction Between an Ellipse and a Hyperbola in Terms of Their Control Points
Show that a rational quadratic Bézier curve, which parametrizes an ellipse (hyperbola), intersects the line segment from the control point \( c_1 \) to \( (c_0 + c_2)/2 \) after (before) the midpoint.

Problem 3.4.2 Parametrization of a Segment of a Hyperbola
Determine a rational quadratic parametrization of the segment of a hyperbola, defined implicitly by \( xy = 1, \ 0 < x, y \leq a \).

**Answer**
middle weight of a standard parametrization for \( a = 2 \):

Problem 3.4.3 Program: Matrix Representation of a Rational Quadratic Bézier Curve
Write a program which determines the matrix representation
\[
v^\top Av = 0, \quad v = (p_1, p_2 \mid q)^\top
\]
of a rational quadratic Bézier parametrization \( (p_1/q, p_2/q) \) with control points \( C \) and weights \( w \).

**Answer**
\[
\max_{j,k} |a_{j,k}| / \min_{j,k} |a_{j,k}| \text{ for } C = [10; 00; 01] \text{ and } w = [1; 1/2; 1]:
\]
Problem 3.4.4 Implicit Form of a Rational Quadratic Bézier Curve
Determine the implicit form of the conic section corresponding to the rational quadratic Bézier curve with
\[
(C|w) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1/2 \\ 0 & 2 & 1 \end{pmatrix}.
\]

Answer
quotient of the coefficients of \(x^2\) and \(y^2\) in the implicit equation:

\[
\frac{\text{coefficient of } x^2}{\text{coefficient of } y^2}.
\]

Problem 3.4.5 Parametrizations of a Hyperbola
Determine all rational quadratic parametrizations of the hyperbola \(Q : x_1x_2 = 1\).
4 B-Splines

4.1 Recurrence Relation

Problem 4.1.1 Polynomial Segments of Quadratic and Quartic B-Splines
Determine the polynomial segments of the B-splines with the knot vectors

\[ (0,0,1,1), \quad (0,1,1,2), \quad (0,0,1,1,2,2). \]

Answer
values of the B-splines at \( x = 1/2 \): a) \( \square \), b) \( \square \), c) \( \square \)

Problem 4.1.2 Program: B-Splines with Support on a Common Knot Interval
Write a program which determines for all B-splines \( \mathbf{b}_{n,k,\xi} \) corresponding to a knot sequence \( \xi_0, \ldots, \xi_{2n+1} \) the monomial form \( p_{k,0} + p_{k,1}x + \cdots + p_{k,n}x^n \) of the polynomial segments on the common knot interval \([\xi_n, \xi_{n+1}]\).

Answer
largest coefficient of the monomials \( x^k \) for the knots 0, 1, 3, 6, 10, 15: \( \square \)

Problem 4.1.3 B-Spline, defined via Cross Sections of a Cube

Denote by \( b(t) \) the area of the intersection of the plane \( H : x + y + z = t \) with the unit cube \([0, 1]^3\).
Show that \( b \) is a multiple of a B-spline.

Answer
quotient of \( b \) and the appropriate B-spline: \( \square \)

Problem 4.1.4 Integral of a Uniform B-Spline
Show that \( \int_\mathbb{R} b^n = 1 \).

4.2 Differentiation

Problem 4.2.1 Third Derivative of a Cubic B-Spline
Determine the third derivative of the B-spline with knots 0, 1, 3, 6, 10.

Answer
largest value of the third derivative: \( \square \)
4.3 Representation of Polynomials

Problem 4.3.1 Spline with Polynomial Coefficients
Which polynomial does the spline
\[ \sum_{k \in \mathbb{Z}} k^2 b^2(x - k) \]
represent?

**Answer**
sum of the coefficients of the monomials for \( p \):  

Problem 4.3.2 Marsden’s Identity for Bernstein Polynomials
Specialize Marsden’s identity for the interval \([0, 1]\) and \((n + 1)\)-fold knots at 0 and 1. Give a direct proof of the resulting formula for Bernstein polynomials.

**Answer**
largest Marsden coefficient for \((x - 1/2)^3\):  

Problem 4.3.3 Program: Representation of Polynomials
Write a program which determines the coefficients \( q(k) = q_1 + q_2 k + \cdots \) in the representation of a polynomial \( p \) of degree \( \leq n \) by the B-splines \( b^n_{k, \xi} \) with uniform knots \( \xi_k = k \in \mathbb{Z} \).

**Answer**
largest monomial coefficient \( q_k \) for \( p(x) = 1 + x + \cdots + x^5 \):  

4.4 Splines

Problem 4.4.1 Basis of Truncated Powers for Simple Knots
Show: For simple knots \( \xi_0, \ldots, \xi_{m+n} \), the monomials and the so-called truncated powers,
\[ 1, x, \ldots, x^n, \phi_k(x) = (\max(0, x - \xi_k))^n, \quad k = n+1, \ldots, m-1, \]
form a basis for the spline space \( S^n_\xi \).

4.5 Evaluation and Differentiation

Problem 4.5.1 Evaluation of a Cubic Spline
Evaluate the cubic spline \( p \) with coefficients \( c = (2, -4, 5, 5, 2, -2) \) and knots \( \xi = (0, 1, 1, 1, 2, 2, 4, 5, 6, 7) \) at the points \( x = 1, 2, 3, 4 \).

**Answer**
\( p(1) + \cdots + p(4) \):  

Problem 4.5.2 Program: Integrating a Spline
Write a program which determines B-spline coefficients \( d_k \) and a knot sequence \( \eta \) of an indefinite integral of a spline with coefficients \( c_k \) and knots \( \xi_k \).

**Answer**
max \( k \), min \( k \) for a cubic spline with \( \xi_k = 2^k \) and \( c_k = \xi_k, k = 0, \ldots, 5 \):  

Problem 4.5.3 Values of a Uniform Cubic Spline at the Knots
Derive a formula for the values of a uniform cubic spline at the knots.

**Answer**
absolute value of the spline at a knot for \( c_k = (-1)^k \):  

Problem 4.5.4 Program: B-Spline Coefficients and Knots of a Cubic Spline from Hermite Data
Write a program which determines the B-spline coefficients $c_k$ and a knot sequence $\xi$ of a cubic spline from the values $p_k$ and the derivatives $d_k$ at the double knots $x_k$.

Answer
largest coefficient $c_k$ for $x_k = p_k = d_k = k^2$, $k = 0, \ldots, 5$: 

Problem 4.5.5 Integral of a B-spline
Show that

$$\int_{\mathbb{R}} b^n_{k, \xi} = \frac{\xi_{k+n+1} - \xi_k}{n+1}.$$ 

Problem 4.5.6 Program: Hermite Data from B-Spline Coefficients
Write a program which computes the Hermite data $p(\xi_\ell)$, $p'(\xi_\ell)$ from the B-spline coefficients of a cubic spline $p \in S^3_\xi$ with double knots.

Answer
largest value of $p(\xi_\ell)$ and $p'(\xi_\ell)$ for $\xi_{2k} = \xi_{2k+1} = k^2$ and $c_k = (-1)^k$, $k = 0, \ldots, 5$: 

Problem 4.5.7 Differentiation of a Cubic Spline
Compute the B-spline coefficients of the first and second derivative of the cubic spline with

$$\xi_0 = 0, 1, 1, 2, 4, 7, 8, 9, 10 = \xi_{10}, \quad c_0 = 0, 3, 5, 9, 9, -1, -6 = c_6,$$

on the parameter interval $D^3_\xi = [1, 7]$.

Answer
smallest B-spline coefficient of the derivatives: 

4.6 Periodic Splines

Problem 4.6.1 Periodic Extension by a Quadratic Spline
Determine the 3-periodic quadratic spline $p$ with simple integer knots $\xi_k = k$ with $p(x) = x$ for $x \in [0, 1]$.

Answer
B-spline coefficient $c_{100}$: 

5 Approximation

5.1 Schoenberg’s Scheme

Problem 5.1.1 Error of the Derivative for Schoenberg’s Scheme
Derive the estimate
\[
\max_x |f'(x) - (Qf)'(x)| \leq h \max_y |f''(y)|, \quad h = \max_{\xi_k \leq x \leq \xi_{k+n}} |\xi_{k+n} - \xi_k|,
\]
for Schoenberg’s scheme of degree \(n\).

Problem 5.1.2 Approximation of Fractional Powers with Schoenberg’s Scheme
Construct knot sequences \(\xi_0, \ldots, \xi_{m+2}\), for which Schoenberg’s scheme with quadratic splines approximates the function \(x^\alpha\) for a given exponent \(\alpha \in (0, 1) \cup (1, 2)\) on the interval \([0, 1]\) with error less than \(c(\alpha) m^{-2}\).

Answer
exponent \(\beta(\alpha)\) for the ansatz \(\xi_\ell = ((\ell - 2)/(m - 2))^{\beta}\) and \(\alpha = 3/2\):

Problem 5.1.3 Error of Schoenberg’s Approximation Applied to the Standard Parabola
Determine the error of Schoenberg’s approximation,
\[
f \approx Qf = \sum_k f(\xi_k^n) b^n_{k,\xi},
\]
to the function \(f(x) = x^2\) for quadratic splines with uniform knots \(\xi_k = kh, \ k \in \mathbb{Z}\).

Answer
absolute value of the error: \(h^2\)

Problem 5.1.4 Convergence of Schoenberg’s Scheme for Continuous Functions
Show that, for a bi-infinite knot sequence \(\xi\) and a continuous function \(f\), Schoenberg’s approximation,
\[
f \approx Qf = \sum_k f(\xi_k^n) b^n_{k,\xi},
\]
converges for any \(x \in \mathbb{R}\), as the maximal length \(h\) of the knot intervals tends to 0.

5.2 Quasi-Interpolation

Problem 5.2.1 Quasi-Interpolant for Quadratic Splines
Construct functionals of the form
\[
Q_k f = \sum_{\nu=0}^2 w_{k,\nu} f(\eta_{k+\nu}), \quad \eta_k = (\xi_k + \xi_{k+1})/2,
\]
for a quasi-interpolant of maximal order with quadratic splines.

Answer
largest weight \(w_{0,\nu}\) for \(\xi_0 = 0, 1, 4, 9 = \xi_3\):
5 Approximation

Problem 5.2.2 Quasi-Interpolant for Uniform Cubic Splines with Double Knots
Determine quasi-interpolant functionals of the form

\[ Q_{2k+\nu} f = \sum_{\mu=0}^{4} w_{\nu,\mu} f(kh + \mu h/2) \]

for cubic splines with double knots \( \xi_{2k+\nu} = kh, \ k \in \mathbb{Z}, \ \nu \in \{0,1\} \).

**Answer**
largest entry of \((w_{2k,0}, \ldots, w_{2k,4})\) for the choice \(w_{2k,4} = 0\):

Problem 5.2.3 Program: Quasi-Interpolant for Uniform Splines Based on Values at Knots
Write a program which computes the coefficients \(w_\nu\) with minimal norm \((\sum_\nu w_\nu^2)^{1/2}\) of functionals

\[ Q_k f = \sum_{\nu=0}^{n+1} w_\nu f(\xi_{k+\nu}) \]

for a quasi-interpolant of maximal order with splines of degree \(\leq n\) with uniform knots \(\xi_\ell = \ell h\).

**Answer**
largest coefficient \(w_\nu\) for \(n = 4\):

5.3 Accuracy of Quasi-Interpolation

5.4 Stability

5.5 Interpolation

Problem 5.5.1 Failure of Diagonal Dominance for Interpolation with Cubic Splines
Show that the matrix

\[ b_{3,k,\xi}(t_j), \quad 1 \leq j, k \leq m, \]

for cubic spline interpolation at the knots \(t_j = \xi_{j+2}\) is, in general, not diagonally dominant.

**Answer**
not diagonally dominant for \(\xi_{j+2} = q^j\) if \(q \geq \)

Problem 5.5.2 Interpolation Matrix for Cubic Splines for the Not-A-Knot Condition
Determine the interpolation matrix for uniform cubic splines with values assigned at the knots, using the not-a-knot boundary condition.

**Answer**
max_\nu |w_\nu| / min_\nu |w_\nu| for the not-a-knot condition \(\sum_\nu w_k c_k = 0\):

Problem 5.5.3 Spline Interpolation of High Order Hermite Data
Show that, for a spline \(p\) of degree \(\leq n\) with simple knots \(\xi_k, \ k \in \mathbb{Z}\), the interpolation problem

\[ p^{(j)}(\xi_{nk}) = f^{(j)}(\xi_{nk}), \quad j = 0, \ldots, n-1, \ k \in \mathbb{Z}, \]

is uniquely solvable.
Problem 5.5.4 Periodic Cubic Lagrange Spline
Determine the 1-periodic cubic Lagrange spline with the knots
\[ \xi_k = k/M, \quad k \in \mathbb{Z}, \]
which equals 1 at \( \xi_0 = 0, \xi_{\pm M} = \pm 1, \ldots \)

Answer
B-spline coefficient \( c_0 \) for \( M = 4: \) }

5.6 Smoothing

Problem 5.6.1 Characterization of the Natural Spline Interpolant as Orthogonal Projection
Show that the second derivative \( p'' \) of the natural spline interpolant of a function \( f \) at the points \( x_0, \ldots, x_M \) is the orthogonal projection of \( f'' \) with respect to the scalar product
\[ \langle \varphi, \psi \rangle = \int_{x_0}^{x_M} \varphi \psi \]
onoonto piecewise linear functions which vanish at \( x_0 \) and \( x_M \).

Problem 5.6.2 Slalom Spline
Show that among all functions \( f \) with
\[ \alpha_i \leq f(x_i) \leq \beta_i, \quad i = 0, \ldots, M, \]
the natural cubic spline with knots at \( x_0 < \cdots < x_M \) minimizes \( \int_{x_0}^{x_M} |f''|^2 \).

Problem 5.6.3 Limit of the Smoothing Spline as a Weight Tends to Infinity
Prove that the smoothing spline tends to \( f_k \) at the knot \( x_k \) if the weight \( w_k \) tends to infinity.
6 Spline Curves

6.1 Control Polygon

Problem 6.1.1 Approximating Greek Letters with Spline Curves
Construct spline curves which qualitatively correctly describe the Greek letters α, β, and γ.

Use as few control points as possible.

Problem 6.1.2 Program: Bounding Box Test for Spline Curves
Write a program which tests with the aid of the bounding boxes if a point can lie on a spline curve.

6.2 Basic Properties

Problem 6.2.1 Distance of a Closed Cubic Spline Curve to its Control Polygon
Estimate the distance in the maximum norm of the closed cubic spline curve $p$ with uniform knots and control points

$$C = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}^t$$

to its control polygon.

Answer
bound provided by the standard estimate:

Problem 6.2.2 Distance of a Bézier Curve to its Control Polygon
Specialize the estimate of the distance between a spline curve and its control polygon to the case of Bézier curves.

Answer
$$\|p(t) - c(t)\|_\infty \leq \max_k \|\Delta^2 c_k\|_\infty$$

Problem 6.2.3 Distance of a Uniform Spline Curve from a Regular $m$-gonal Control Polygon
Estimate the distance of the closed spline curves of degree $\leq n$ with uniform knots and control points

$$c_k = (\cos(2\pi k/m), \sin(2\pi k/m))$$

with $m \geq 3$ from their control polygons.

Answer
$$\|p(t) - c(t)\|_\infty \leq \frac{(n + 1)}{m^2}$$
6.3 Refinement

Problem 6.3.1 Refinement of a B-Spline
Express the B-spline with knots 0, 1, 2, 4 as a linear combination of standard uniform B-splines.

**Answer**
smallest coefficient of the uniform B-splines:

Problem 6.3.2 Knot Insertion for a Spline Curve
For a cubic spline curve with knots \( \tau_k = k, \) \( k = 0, \ldots, 9, \) and control points
\[
C = \begin{pmatrix}
0 & 1 & 0 & 2 & 2 & 3 \\
0 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}^t,
\]
insert the knots 3.5 and 6.

**Answer**
largest coordinate of the newly generated control points:

Problem 6.3.3 Convergence of Subdivision for Uniform Splines
Show that the control polygons \( c^0, c^1, \ldots, \) generated by the subdivision algorithm for uniform splines, converge to the spline curve:
\[
\| c^m(t) - p(t) \|_\infty = O(4^{-m})
\]
for all parameters \( t. \)

Problem 6.3.4 Knot Removal for a Cubic Spline Curve
Which of the knots \( \tau_k = k, \) \( k = 0, \ldots, 9, \) of the cubic spline curve with control points
\[
C = \begin{pmatrix}
0 & 4 & 1 & 3 & 6 & 6 \\
0 & 0 & 3 & 4 & 3 & 0
\end{pmatrix}^t
\]
can be removed, and what are the control points corresponding to the coarser knot sequence?

**Answer**
removable knot:

Problem 6.3.5 Simultaneous Knot Insertion for Uniform Cubic Splines with Double Knots
Derive a formula for simultaneous knot insertion for cubic spline curves with double knots:
\[
\tau : \ldots, 0, 0, h, h, \ldots \rightarrow \tau' : \ldots, 0, 0, h/2, h/2, \ldots
\]

**Answer**
largest weight in the formula \( c'_k = \sum_j w_{k,j}c_j \) for the new control points:

Problem 6.3.6 Knot Insertion for Cubic Splines with Double Knots
Insert the knot 5 twice for the cubic spline curve with
\[
C = \begin{pmatrix}
0 & 0 & 1 & 1 & 2 & 2 \\
0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix}^t
\]
and \( \tau_0 = 0, 2, 4, 6, 8, 9 = \tau_9. \)

**Answer**
largest coordinate of the newly generated control points:
Problem 6.3.7 Knot Insertion for a Quartic Closed Spline Curve
Insert the knot 2 for the quartic closed spline curve with
\[ c_0 = (1, 0), \ c_1 = (0, 1), \ c_2 = (0, 0), \ \tau_0 = 0, \ \tau_1 = 1, \ \tau_2 = 3, \]
and periodicity interval [0, 4].
\textbf{Answer}
largest coordinate of the newly-generated control points:

6.4 Algorithms

Problem 6.4.1 Point on a Cubic Spline Curve and Tangent Vector
Determine \( p(3) \) and \( p'(3) \) for the cubic spline curve \( p \) with control points
\[ C = \begin{pmatrix} -8 & -5 & 5 & 8 & -2 & -6 \\ 2 & -9 & 6 & -9 & -5 \end{pmatrix}^t \]
and knots 0, 1, 1, 2, 5, 7, 7, 7, 8.
\textbf{Answer}
largest coordinate of \( p(3) \) and \( p'(3) \):

Problem 6.4.2 Curvature of a Cubic Spline Curve
Derive a formula for the curvature of a uniform cubic spline curve in \( \mathbb{R}^3 \) at the knots. Give
a geometric interpretation. As an example, consider the closed cubic spline curve with the
vertices of the unit square \([0, 1]^2\) as control points.
\textbf{Answer}
curvature at the knots for the example:

Problem 6.4.3 Bézier Form of a Quadratic Spline Curve
Determine the Bézier form of the closed quadratic spline curve with control points
\[ c_0 = (0, 0), \ (0, -1), \ (1, 0), \ (0, 1), \ (-1, 0) = c_4 \]
and knots \( \tau_0 = 0, \ 1, \ 1, \ 3, \ 6 = \tau_4 \) in the periodicity interval [0, 10].
\textbf{Answer}
largest coordinate of the newly-generated control points:

Problem 6.4.4 Program: Variation of a Control Polygon with respect to a Hyperplane
Write a program which computes the variation of a control polygon with respect to a hyperplane \( H \subset \mathbb{R}^d \), determined by \( d \) points.

Problem 6.4.5 Bézier Representation of a Quartic Spline Curve
Determine the Bézier representation of the quartic spline curve with control points
\[ C = \begin{pmatrix} 0 & 12 & 12 & 0 & 0 \\ 0 & 0 & 12 & 12 & 0 \end{pmatrix}^t \]
and uniform knots.
\textbf{Answer}
largest coordinate of the Bézier control points:

6.5 Interpolation
7 Multivariate Splines

7.1 Polynomials

Problem 7.1.1 Representation of a Bilinear Polynomial in Terms of Bernstein Polynomials
Express the polynomial \( p(x) = 1 + 2x_1 + 3x_2 + 4x_1x_2 \) as a linear combination of bivariate Bernstein polynomials of degree \((2, 2)\).

Answer
largest Bernstein coefficient: [ ]

7.2 Polynomial Approximation

Problem 7.2.1 Zeros of the Error of the Polynomial Orthogonal Projection
Prove that the error \( P^n f - f \) of the orthogonal projection onto \( \mathbb{P}^n[0, 1] \) has at least \( n + 1 \) simple zeros in \((0, 1)\). Show that this implies

\[
\| P^n f - f \|_{\infty, [0, 1]} \leq \| f^{(n+1)} \|_{\infty, [0, 1]}
\]

for any smooth function \( f \).

7.3 Splines

Problem 7.3.1 Program: Values of Multivariate B-Splines
Write a program which generates the \( d \)-dimensional array \( b_{n}(k), 1 \leq k \leq n \), of values of a uniform multivariate B-spline at the knots.

Answer
\( b(4,7,9)(3,2,5) = [ ] \)

Problem 7.3.2 Nonexistence of B-Splines of Minimal Total Degree
Show that, on a bivariate tensor product grid, bivariate B-splines of total degree \( \leq n \) and smoothness \( n - 1 \) do not exist for \( n > 1 \). More precisely, any function with continuous partial derivatives of order \( \leq n - 1 \), which is a polynomial of total degree \( \leq n \) on each grid cell \( k+[0, 1]^2 \), \( k \in \mathbb{Z}^2 \), and has support in a square \([0,m]^2\), must vanish identically.

7.4 Algorithms

Problem 7.4.1 Gradient of a Bivariate Biquadratic Spline
Compute the gradient of the spline \( p = \sum_k c_k b_{k, \xi}^{(2,2)} \) with coefficients

\[
\begin{array}{ccc}
-8 & 4 & -8 \\
0 & -4 & 0 \\
8 & 4 & -8
\end{array} = c_{(0,0)}
\]

and uniform knots \( \xi_{\nu, \ell} = \ell h \) at \((h/2, h/2)\).

Answer
largest coordinate of the gradient for \( h = 1/2 \): [ ]
Problem 7.4.2 Evaluation of a Bivariate Uniform Spline
Evaluate the uniform bivariate spline 
\[ p(x) = \sum_k c_k b^{(3,2)}(x - k) \] with coefficients
\[
\begin{align*}
  c_{(0,0)} &= 0 & 0 \\
  \ldots & 15 & 9 & \ldots \\
  \ldots & 20 & 4 & \ldots \\
  30 & 18 &= c_{(3,1)} \\
\end{align*}
\]
at \( x = (3.5, 2) \).
Answer
\[ p(3.5, 2): \boxed{\ldots} \]

7.5 Approximation Methods

Problem 7.5.1 Program: Evaluation of a Multivariate Polynomial
Write a program which evaluates a multivariate polynomial
\[ p(x) = \sum_{k_1=0}^{n_1} \ldots \sum_{k_d=0}^{n_d} c_k x^k \]
at the grid of points \( x = (x_1,j_1,\ldots,x_d,j_d), 0 \leq j_\nu \leq n_\nu \).
Answer
largest value of \( p(x) \) for \( n = (3, 3, 3) \), \( x_\nu = [1 : 4] \), and \( c_k = 1 + k_1 + k_2 + k_3: \boxed{\ldots} \)

Problem 7.5.2 Program: Multivariate Polynomial Interpolation
Write a program which computes the interpolating polynomial of coordinate degree \((n_1, \ldots, n_d)\) to data \( f(j_1,\ldots,j_d) \) on a tensor product grid \((x_1,j_1,\ldots,x_d,j_d), 0 \leq j_\nu \leq n_\nu \).
Answer
largest monomial coefficient for \( d = 4, n_\nu = 3, x_\nu,j = j, \) and \( f_j = (-1)^{j_1+j_2+j_3+j_4}: \boxed{\ldots} \)

7.6 Hierarchical Bases

Problem 7.6.1 Dimension of a Hierarchical Spline Space
Determine the dimension of the bilinear hierarchical spline space on \( D = [4, 36] \times [4, 28] \), determined by the tree
\[ \Xi: \xi^* \to \eta, \tilde{\eta}, \eta \to \zeta \]
where
\[
\begin{align*}
  \xi_1^* &: 0, 4, \ldots, 40, \\
  \eta_1 &: 2, 4, \ldots, 20, \\
  \tilde{\eta}_1 &: 24, 26, \ldots, 32, \\
  \zeta_1 &: 8, 9, \ldots, 20, \\
  \xi_2^* &: 0, 4, \ldots, 32, \\
  \eta_2 &: 2, 4, \ldots, 20, \\
  \tilde{\eta}_2 &: 12, 14, \ldots, 24, \\
  \zeta_2 &: 8, 9, \ldots, 18. \\
\end{align*}
\]
Answer
dimension: \boxed{\ldots} 

Problem 7.6.2 Program: Adaptive Approximation of the Square Root with Hierarchical Linear Splines
Construct adaptively a linear hierarchical spline for approximating the function \( x \mapsto \sqrt{x}, x \in [0, 1] \). Use linear interpolation and subdivide a subinterval at the midpoint if the error is larger than tol. Plot the maximal errors on the subintervals of the resulting hierarchical partition
Answer
number of intervals to achieve an error \( \leq 0.001: \boxed{\ldots} \)
8 Surfaces and Solids

8.1 Bézier Surfaces

Problem 8.1.1 Approximation of a Sphere by Bi-Quadratic Bézier Patches
Determine a fully symmetric approximation of the sphere $S : x_1^2 + x_2^2 + x_3^2 = 3$ by 6 biquadratic Bézier patches. The patches should touch the sphere at $(\pm 1, \pm 1, \pm 1)$ (endpoint interpolation) as well as at $(\pm \sqrt{3}, 0, 0), (0, \pm \sqrt{3}, 0), (0, 0, \pm \sqrt{3})$.

Answer

distance of the middle Bézier control points to the origin:  

Problem 8.1.2 Nonexistence of a Tangent Plane for a Bézier Patch with a Degenerate Boundary
Show by example that a Bézier patch with $c_{0,0} = \ldots = c_{n,0}$ does, in general, not have a tangent plane at the multiple boundary control point.

Problem 8.1.3 Conditions for Tangent Plane Continuity
Assume that two regular Bézier parametrizations of degree $(n, n)$ share a common boundary curve:

$$q(1, s) = p(0, s), \quad 0 \leq s \leq 1.$$  

A sufficient condition for tangent plane continuity is

$$\partial_1 q(1, s) = \alpha \partial_1 p(0, s) + (\beta_0 (1 - s) + \beta_1 s) \partial_2 p(0, s), \quad 0 \leq s \leq 1,$$

with $\alpha > 0$. Express this condition in terms of the control points.

8.2 Spline Surfaces

Problem 8.2.1 Bicubic Model of a Torus
Model a torus with radii 3 and 1 with uniform bicubic splines with 16 control points. The surface should touch the torus at least at 16 points.

8.3 Subdivision Surfaces

Problem 8.3.1 Program: 4-Point Scheme for Surfaces
Program the tensor product 4-point scheme, which is based on bicubic interpolation, for a rectangular quadrilateral mesh.

Problem 8.3.2 Limit Points for the Scheme of Catmull-Clark
Determine the limits of the irregular vertices for the Catmull-Clark algorithm applied to the wireframe of the standard cube $[-1, 1]^3$.

Answer

distance of the vertex limits to the origin:  

8.4 Blending

Problem 8.4.1 Coon’s Patch for Quadratic Bézier Boundaries
Blend the boundary values

$$p(x_1, 0) = 0, \quad p(1, x_2) = x_2, \quad p(x_1, 1) = 1, \quad p(0, x_2) = x_2^2, \quad 0 \leq x_1, x_2 \leq 1,$$  

with a biquadratic polynomial in Bernstein-Bézier form.

Answer

middle Bézier coefficient $c_{1,1}$:
8.5 Solids

Problem 8.5.1 Program: Volume of a Bézier Solid

Write a program which determines the volume of a Bézier solid with regular parametrization.

Answer

volume for the solid with control points \( c_k = k(1 + |k|^2), k_{\nu} \leq 2 \): 

9 Finite Elements

9.1 Ritz-Galerkin Approximation

Problem 9.1.1 Ritz–Galerkin Approximation with Sine Functions
Determine the Ritz–Galerkin approximation of the boundary value problem

\[-u'' + u = x, \quad u(0) = u(\pi) = 0,\]

for the finite elements \(x \mapsto \sin(kx), \ k = 1, \ldots, n\).

Answer
coefficient of \(\sin(10x)\):

Problem 9.1.2 Program: Ritz–Galerkin Approximation of a Radially Symmetric Poisson Problem
Write a program which determines the Ritz–Galerkin approximation \(u_n\) of the radially symmetric Poisson problem

\[-\frac{1}{r}(ru')' = \exp(r^2), \quad u(1) = 0,\]
on the unit disc for the basis functions

\(B_1, \ldots, B_n, \quad B_k(r) = 1 - r^{2k}\).

Plot \(u_n\) and determine the residual \(e_n(r) = -(1/r)(ru_n)' - \exp(r^2)\).

Answer
\(\max_{0 \leq r \leq 1} |e_5(r)| = \)

Problem 9.1.3 \(H^1\)-Error of Univariate Hat–Functions
Derive the error estimate

\[|u - u_h|_1 \leq h|u|_2, \quad |v|_k^2 = \int_0^1 |v^{(k)}|^2,\]

for piecewise linear interpolants \(u_h\) of a function \(u \in H^2(0,1)\).

9.2 Weighted B-Splines

Problem 9.2.1 Ritz–Galerkin Integrals of Bilinear B-Splines
Determine the Ritz–Galerkin integrals

\[g_{k,\ell} = \int \text{grad} b^n_{k,h} \text{grad} b^n_{\ell,h}\]

for bivariate tensor product B-splines of degree \(n = (1,1)\).

Answer
\(\sum_{\ell} |g_{k,\ell}| = \)

Problem 9.2.2 Program: Rvachev Operations for Weight Functions Represented by \(m\)-Files
Write a program \(\text{rfct}(w,\text{operation},w_1,w_2)\), which implements Rvachev’s method for weight functions represented by MATLAB \(m\)-files \(w_1, w_2\), by generating an \(m\)-file \(w\) according to the specified Boolean operation (union, intersection, or complement).
9.3 Isogeometric Elements

Problem 9.3.1 Bijectivity of a Bilinear Isoparametric Transformation of the Unit Square
Show that a bilinear isoparametric transformation of the unit square is bijective if and only if the image is a convex quadrilateral.

9.4 Implementation

Problem 9.4.1 Numerical Integration of Bilinear Splines over a Boundary Cell
For $\Omega : 0 < x_1, x_2 < 1$, and bilinear B-splines $b_k$ with grid width 1, determine weights $\gamma_k$ such that

$$\int_\Omega p(x) \, dx = \sum_k \gamma_k c_k, \quad p = \sum_k c_k b_k.$$  

Answer
smallest weight $\gamma_k$: [ ]

Problem 9.4.2 Gauß Parameters for a Bivariate Boundary Cell
Using the univariate formula

$$\int_0^1 f \approx \frac{1}{2} f \left( \frac{1}{2} - \sqrt{3}/6 \right) + \frac{1}{2} f \left( \frac{1}{2} + \sqrt{3}/6 \right),$$

determine weights $\gamma_\ell$ and nodes $(x_\ell, y_\ell)$ for integration over the boundary cell

$$\Omega : 1 - 2xy > 0, \ 0 < x, y < 1.$$  

Answer
approximation of $\int_\Omega x^y dx dy$: [ ]

9.5 Applications

Problem 9.5.1 Strain Tensor for a Radially Symmetric Displacement
Compute the strain tensor $\varepsilon(u)$ for a radially symmetric displacement $u(x) = \varphi(r)x, \ r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

Answer
$\sum_{k,\ell} \varepsilon_{k,\ell}$ for $\varphi(r) = r^2$ and $x = (1, 1, 1)$: [ ]

Problem 9.5.2 Program: Elasticity Bilinear Form for Hat-Functions on a Tetrahedron
Write a program which computes the $4 \times 4$ matrix of $3 \times 3$ blocks

$$\int_{[p_1,p_2,p_3,p_4]} \sigma(B_k e_\alpha) : \varepsilon(B_\ell e_\beta)$$

for the hat-functions $B_j$, which correspond to the vertices $p_j$ of a tetrahedron, and the unit vectors $e_1, e_2, e_3$.

Answer
largest matrix entry for $p_{j,k} = j^k$, $1 \leq j \leq 4$, $1 \leq k \leq 3$: [ ]
Part II
Hints
1 Polynomials

1.1 Monomial Form

Problem 1.1.1 Derive a recursion for the summands.

1.2 Taylor Approximation

Problem 1.2.1 Determine the worst case for each factor of the Taylor remainder.

Problem 1.2.2 Determine first an upper bound \( r(h) \) for the Taylor remainder. Then solve the equation \( r(h) = 10^{-12} \) for \( h \).

1.3 Interpolation

Problem 1.3.1 Use the Lagrange form of the interpolating parabola.

Problem 1.3.2 Use the Lagrange form of the cubic interpolant.

Problem 1.3.3 Multiplication of a polynomial by \((x + c)\) corresponds to the operation \(a \rightarrow [0; a] + c \ast [a; 0]\) on the coefficients \(a(k)\).

1.4 Bernstein Polynomials

Problem 1.4.1 Use the matrix describing the basis transformation.

Problem 1.4.2 Observe that the product of two Bernstein polynomials equals, up to a constant factor, a Bernstein polynomial of higher degree.

1.5 Properties of Bernstein Polynomials

Problem 1.5.1 Use the formula for \((b^n_k)'\).

Problem 1.5.2 It suffices to consider quadratic polynomials.

1.6 Hermite Interpolant

Problem 1.6.1 Apply the formula for the Bernstein coefficients. Use the symmetry of the first two segments.

Problem 1.6.2 Represent the interpolant in Bernstein form.

1.7 Approximation of Continuous Functions

Problem 1.7.1 Use the program `bernstein` to evaluate the Bernstein polynomials.

Problem 1.7.2 Show first that \( p'_n = n \sum_{k=0}^{n-1} (f((k + 1)/n) - f(k/n))b^n_{k-1} \) and apply the mean value theorem.
2 Bézier Curves

2.1 Control Polygon

Problem 2.1.1 Convert to monomial form \( a_0 + a_1 t + a_2 t^2 \) and substitute \( s = t + \alpha \) to obtain the standard parametrization of a parabola.

Problem 2.1.2 Write the components \( p_\nu \) of the Bézier parametrization in monomial form and consider the equation \( p_1^2 + p_2^2 = 1 \).

2.2 Properties of Bézier Curves

Problem 2.2.1 Show that the Bézier parametrization has the form \( p(t) = c_0 + \gamma(c_k - c_0)t^k + O(t^{k+1}) \) and substitute \( s = t^k \).

Problem 2.2.2 Use symmetry with respect to the vertical axis and determine the middle control points by testing the parametrization at \( t = 1/2 \).

2.3 Algorithm of de Casteljau

Problem 2.3.1 Recall that the edges of the control polygon are divided in the ratio \( (1 - t) : t \).

2.4 Differentiation

Problem 2.4.1 Setting the derivatives at \( t = 0 \) of \( y(t) - \exp(x(t)) \) to zero yields nonlinear equations for the derivatives \( x^{(k)}(0) \) and \( y^{(k)}(0) \).

Problem 2.4.2 Use the MATLAB functions \texttt{polyfit} and \texttt{roots}.

2.5 Curvature

Problem 2.5.1 Use subdivision and the formula for the curvature at the endpoints of a Bézier curve.

Problem 2.5.2 Use the endpoint interpolation property and the formula for the curvature at the endpoints.

Problem 2.5.3 Use the endpoint interpolation property and the formula for the curvature at the endpoints.

2.6 Subdivision

Problem 2.6.1 View the Bézier curve as the left part of its extension and traverse de Casteljau’s triangle from left to right, i.e., determine \( p_m^0, \ldots, p_m^{n-m} \) from \( p_{m-1}^0, \ldots, p_{m-1}^{n-m+1} \).

Problem 2.6.2 Show first that the maximal edge length of the polygons in de Casteljau’s algorithm is nonincreasing.

2.7 Geometric Hermite Interpolation

Problem 2.7.1 Use the parametrization \( t \mapsto (\cos t, 2 \sin t) \) to compute the curvature of the ellipse and note that symmetry is preserved by the interpolant.
3 Rational Bézier Curves

3.1 Control Polygon and Weights

Problem 3.1.1 Write a point \( r(t) \) on a quadratic Bézier curve as repeated convex combination of the control points:

\[
r(t) = (1 - \alpha)((1 - \beta)c_0 + \beta c_2) + \alpha c_1.
\]

Problem 3.1.2 Divide numerator and denominator of the rational parametrization by \( w_0 \) and write

\[
w_\ell = (w_\ell / w_{\ell-1})(w_{\ell-1} / w_{\ell-2}) \ldots (w_1 / w_0).
\]

Problem 3.1.3 Note that the middle and right curve can be obtained from the left curve by a rotation and shear transformation, respectively.

3.2 Basic Properties

Problem 3.2.1 Use endpoint interpolation and that, by symmetry, \( c_{1,2} = c_{2,2} \) and \( w_1 = w_2 \).

Problem 3.2.2 Divide numerator and denominator of the parametrization by \( \lambda \) and note that \( w_\ell / \lambda \to 0 \) for \( j,k \neq \ell \).

Problem 3.2.3 Transform the weights to standard form.

3.3 Algorithms

Problem 3.3.1 Apply de Casteljau’s algorithm to the homogeneous control points.

Problem 3.3.2 No hint available for this problem.

3.4 Conic Sections

Problem 3.4.1 Write \( r(t) = (1 - \alpha)c_1 + \alpha(c_0 + c_2)/2 \) and determine \( \alpha \) in terms of the weights \( w_k \).

Problem 3.4.2 Use endpoint interpolation to determine the control points \( c_k \) and determine the middle weight by testing the parametrization in standard form at \( t = 1/2 \).

Problem 3.4.3 Test the equation \( (p_1(t), p_2(t) | q(t))A(p_1(t), p_2(t) | q(t))^t = 0 \) at 5 points \( t \) to obtain a linear system for the matrix entries \( a_{j,k} = a_{k,j} \).

Problem 3.4.4 Compare the coefficients of \( t^\ell \) in the equation \( (p_1, p_2 | q)A(p_1, p_2, q)^t = 0 \) where \( p_\nu(t)/q(t) \) are the components of the rational parametrization.

Problem 3.4.5 Write the polynomials \( p_1, p_2, \) and \( q \) of the rational parametrization as products of their linear factors.
4 B-Splines

4.1 Recurrence Relation

**Problem 4.1.1** Recall that a B-spline with just two different knots coincides with a Bernstein polynomial and use the formula for a B-spline on the first knot interval as well as the recurrence relation.

**Problem 4.1.2** Use the B-spline recursion and note that multiplication with a linear function \( a_1 + a_2 x \) changes the coefficients of a polynomial \( p_1 + p_2 x + \cdots \) according to \( p \rightarrow [a_1 * p; 0] + [0; a_2 * p] \).

**Problem 4.1.3** Split the intersection into triangles determined by the intersections of \( H \) with the edges of the cube.

**Problem 4.1.4** Use induction and the definition of the uniform B-spline via averaging.

4.2 Differentiation

**Problem 4.2.1** Use the recursion for B-spline derivatives.

4.3 Representation of Polynomials

**Problem 4.3.1** Combine the representations of the monomials, obtained from Marsden’s identity, in a suitable way.

**Problem 4.3.2** Recall that the Bernstein polynomial \( b^n_k \) coincides on \([0, 1)\) with the B-spline with an \((n + 1) - k\)-fold knot at 0 and a \((k + 1)\)-fold at 1.

**Problem 4.3.3** Write \( p(x) \) as linear combination of \((x - j)^n\) and \( q(k) \) as linear combination of \( \psi^n_k(j) \).

4.4 Splines

**Problem 4.4.1** Since the number of basis functions match, it suffices to show the linear independence of the monomials and truncated powers.

4.5 Evaluation and Differentiation

**Problem 4.5.1** Apply the de Boor algorithm.

**Problem 4.5.2** Invert the recursion for the coefficients of the derivative of a spline.

**Problem 4.5.3** Use symmetry and that the B-spline values at the knots sum to one.

**Problem 4.5.4** Note that for each point \( x_k \) only two B-splines are relevant. Express \( p_k \) and \( d_k \) in terms of the B-spline coefficients via the explicit formulas for the first and last B-spline segments and solve the resulting local linear systems.

**Problem 4.5.5** Using the formula for differentiating a spline, derive an identity for \( \int_{\xi_k}^x b^n_k,\xi \) and then let \( x \) tend to infinity.

**Problem 4.5.6** Use the formula for differentiating a spline and the de Boor algorithm.

**Problem 4.5.7** No hint available for this problem.

4.6 Periodic Splines

**Problem 4.6.1** Use Marsden’s identity to obtain the relevant coefficients for \([0, 1]\) and apply the periodicity conditions.
5 Approximation

5.1 Schoenberg’s Scheme

Problem 5.1.1 Show that \((Qf)' = \sum_k f'(x_k)b_{n-k,\xi}' with x_k \in [\xi^n_{k-1}, \xi^n_k].\)

Problem 5.1.2 Generalize the arguments for \(\sqrt{x}\) using the ansatz \(\xi_\ell = ((\ell - 2)/(m - 2))\beta \) with an appropriate choice \(\beta(\alpha).\)

Problem 5.1.3 Compare with the exact representation of \(x^2\) via Marsden’s identity.

Problem 5.1.4 Write \(f(x) = \sum_k f(x)b_{n-k,\xi}(x)\) and estimate \(f(x) - f(\xi^n_k)\) using the uniform continuity of \(f\) on compact intervals.

5.2 Quasi-Interpolation

Problem 5.2.1 Test the identity \(Q_k(\cdot - y)^2 = \psi_k(y)\) for \(y = \eta_{k+\nu}..\)

Problem 5.2.2 By symmetry, it is sufficient to construct \(Q_0.\) Test the identity \(Q_0(\cdot - y)^3 = \psi_0(y)\) for \(y = 0, h/2, \ldots, 2h.\)

Problem 5.2.3 Testing the identity \(Q_k(\cdot - y)^n = \psi_k(y)\) for \(y = (k + \mu)h\) leads to an underdetermined linear system which is independent of \(h\) and \(k.\)

5.3 Accuracy of Quasi-Interpolation

5.4 Stability

5.5 Interpolation

Problem 5.5.1 Choose \(\xi_{j+2} = q^j.\) To determine the matrix entries use the formulas for the first and last B-spline segment as well as the fact that the B-splines sum to one.

Problem 5.5.2 Use the formulas for the first and last B-spline segment and the fact that the B-splines sum to one. The not-a-knot condition yields two additional rows of the interpolation matrix which can be determined with the aid of the formulas for differentiating B-splines.

Problem 5.5.3 Note that for each interpolation point exactly \(n\) B-splines are relevant and that these B-splines vanish at all other points.

Problem 5.5.4 Use the ansatz \(c_{k-2} = \gamma(\lambda^k + \lambda^{M-k})\) for the B-spline coefficients of the Lagrange spline.

5.6 Smoothing

Problem 5.6.1 Derive the orthogonality relation \(\langle f'' - p'', b_{n-k,\xi}' \rangle = 0\) which characterizes the orthogonal projection.

Problem 5.6.2 Consider the natural spline interpolants of a minimizing sequence \(f_\ell.\)

Problem 5.6.3 Assume that \(|f_k - p^f(x_k)| \geq \varepsilon\) for a sequence of smoothing splines with weights \(w_k^f \to \infty\) and compare \(E(p^f, \sigma)\) with the value for the interpolating natural spline.
6 Spline Curves

6.1 Control Polygon

Problem 6.1.1 Use multiple knots to model the corners in the letters \( \beta \) and \( \gamma \).

Problem 6.1.2 No hint available for this problem.

6.2 Basic Properties

Problem 6.2.1 No hint available for this problem.

Problem 6.2.2 The Bernstein polynomials \( b^n_k \) correspond to the knot sequence \( \tau_0 = 0, \ldots, 0 = \tau_n < \tau_{n+1} = 1, \ldots, 1 = \tau_{2n+1} \).

Problem 6.2.3 No hint available for this problem.

6.3 Refinement

Problem 6.3.1 Use the formulas of the B-splines on their first and last segment.

Problem 6.3.2 No hint available for this problem.

Problem 6.3.3 Use the estimate for the distance of a uniform spline curve from its control polygon in conjunction with the stability of the B-spline basis.

Problem 6.3.4 Compute the control points of the piecewise constant third derivative of the parametrization.

Problem 6.3.5 Insert successively the knots \( h/2, 3h/2, 5h/2, \ldots \), and then simultaneously double their multiplicity.

Problem 6.3.6 No hint available for this problem.

Problem 6.3.7 Form a sufficiently large periodic extension of the control points.

6.4 Algorithms

Problem 6.4.1 No hint available for this problem.

Problem 6.4.2 Derive first a formula for \( p'(\tau_k) \) and \( p''(\tau_k) \) in terms of the control points \( c_{k-3}, c_{k-2}, \) and \( c_{k-1} \), and then substitute the resulting expressions into the definition of curvature.

Problem 6.4.3 No hint available for this problem.

Problem 6.4.4 Determine first the equation of the hyperplane by solving an appropriate linear system.

Problem 6.4.5 No hint available for this problem.

6.5 Interpolation
7 Multivariate Splines

7.1 Polynomials

Problem 7.1.1  Use the formulas for the derivatives at $x = (0, 0)$.

7.2 Polynomial Approximation

Problem 7.2.1  Use that $\int (e - \varepsilon q)^2 < \int e^2$ for small $\varepsilon$ if $q$ has the same sign as the error $e$.

7.3 Splines

Problem 7.3.1  Use a subroutine which generates the values of the uniform univariate B-spline via the recurrence relation.

Problem 7.3.2  Consider the piecewise polynomial function restricted to lines parallel to $(1, -1)$. Use the minimal support property of B-splines to conclude that univariate splines with support on a single grid interval must vanish identically.

7.4 Algorithms

Problem 7.4.1  No hint available for this problem.

Problem 7.4.2  No hint available for this problem.

7.5 Approximation Methods

Problem 7.5.1  Process the array $C$ successively in each of the coordinate directions.

Problem 7.5.2  Solve successively the univariate interpolation problems in each of the coordinate directions.

7.6 Hierarchical Bases

Problem 7.6.1  Subtract from the number of B-splines corresponding to a knot sequence in the tree $\Xi$ the number of B-splines representable on finer grids.

Problem 7.6.2  Derive an explicit formula for the error of the linear interpolant.
8 Surfaces and Solids

8.1 Bézier Surfaces

Problem 8.1.1  By symmetry, the Bézier control points \( c_{0,1}, c_{1,0}, c_{1,2}, c_{2,1} \) lie on the planes through the corresponding cube edges and the origin and can be determined using the endpoint interpolation property.

Problem 8.1.2  Construct a biquadratic patch containing three different line segments emerging from the origin.

Problem 8.1.3  Express the partial derivatives in terms of differences of the Bézier control points.

8.2 Spline Surfaces

Problem 8.2.1  The control points for approximating a circle form a square. Determine its diameter and rotate its position by the angles \( \pi/2, \pi \) and \( 3\pi/2 \).

8.3 Subdivision Surfaces

Problem 8.3.1  Process rows and columns of the rectangular point array with the univariate scheme. Note that after each step the array boundaries have to be discarded.

Problem 8.3.2  For an irregular vertex \( p \) denote by \( e_{v} \) the neighboring vertices on edges emerging from \( p \) and by \( v_{v} \) the remaining vertices of the faces containing \( p \). Set \( E = \sum e_{v}, V = \sum v_{v} \) and determine the \( 3 \times 3 \) matrix describing the modification of \( (p, E, V) \) by a subdivision step. The limit can now be determined by an eigenvalue analysis.

8.4 Blending

Problem 8.4.1  No hint available for this problem.

8.5 Solids

Problem 8.5.1  Use the program \texttt{gausspar} to generate a grid of Gauß points and evaluate the parametrization with the program \texttt{splineSolid}.
9 Finite Elements

9.1 Ritz-Galerkin Approximation

Problem 9.1.1 Use the orthogonality of the basis functions and their derivatives:
\[
\int_0^\pi \sin(jx) \sin(kx) \, dx = \int_0^\pi \cos(jx) \cos(kx) \, dx = \frac{\pi}{2} \delta_{j,k}
\]
for 0 < j, k.

Problem 9.1.2 In deriving the variational equations note that \( \int_D \ldots = 2\pi \int_0^1 \ldots r \, dr \) for the unit disc D. Moreover, observe that the integrals \( \int_0^1 \exp(r^2) B_k(r) \, r \, dr \) can be computed recursively.

Problem 9.1.3 By the mean value theorem, there exists a point \( x_k \) in every interval \( D_k = kh + [0, h] \), such that \( u'(x_k) - u'_h(x_k) = 0 \). Use this fact to estimate \( \int_{D_k} |u' - u'_h|^2 \), noting that \( u'_h \) is constant on \( D_k \) with vanishing second order derivative.

9.2 Weighted B-Splines

Problem 9.2.1 Use the values of the scalar products of univariate B-splines and their derivatives. Moreover, note that \( \sum_{\ell} g_{k,\ell} = 0 \).

Problem 9.2.2 Use the MATLAB commands `switch`, `strcat`, `fprintf`, `fopen`, and `fclose`.

9.3 Isogeometric Elements

Problem 9.3.1 Consider the images of the line segments \( \{x\} \times [0, 1] \).

9.4 Implementation

Problem 9.4.1 The relevant bilinear B-splines are \( b_k, k \in \{-1, 0\}^2 \).

Problem 9.4.2 Determine the intersections of \( \Gamma : 1 - 2xy = 0 \) with the boundary of \( (0, 1)^2 \), and partition \( \Omega \) accordingly. Map the tensor product Gauss formula to the resulting subdomains.

9.5 Applications

Problem 9.5.1 By the chain rule, \( \partial_\nu r = x_\nu / r \).

Problem 9.5.2 The integrands are constants which can be determined from the gradients of the hat-functions. Compute the gradients by considering appropriate directional derivatives.