Preface

This book is for people interested in the numerical solution of PDEs, with an emphasis on high-order schemes for hyperbolic problems. In fact, evolutive linear and nonlinear first-order PDEs appear in a variety of applications ranging from geophysical and atmospheric problems to control theory, Fluid Dynamics, and image processing. Although every application has its own features, a usual requirement for the numerical solution is to keep accuracy and stability without overly restrictive assumptions on the time step, which usually imply a high computational complexity.

We will describe and analyze basic and recent developments of the Semi-Lagrangian technique for the approximation of first-order PDEs with a special focus on Hamilton–Jacobi equations. The main purpose is to obtain methods which are unconditionally stable with respect to the choice of the time step. The basic idea dates back to the method proposed by Courant, Isaacson, and Rees in 1952 [CIR52] for first-order systems of linear equations and has gone through a number of improvements and extensions since. In particular, the main developments have been proposed within the Numerical Weather Prediction community [SC91], where this class of schemes has become of common use.

As for the advection equation, the driving force of this method is the method of characteristics which accounts for the flow of information in the model equation. At the numerical level, the Semi-Lagrangian approximation mimics the method of characteristics looking for the foot of the characteristic curve passing through every node and following this curve for a single time step.

In order to derive a numerical method from this general idea, several ingredients should be put together, mainly a technique for ODEs to track characteristics and a reconstruction technique to recover pointwise values of the numerical solution. In this framework, the order of space and time discretizations is typically reversed with respect to more conventional schemes—the first discretization is done in time, the second in space—although the final result is clearly a fully discrete scheme. This reversal is the key to getting weaker stability requirements.

This book has two main purposes. The first is to present Semi-Lagrangian approximation schemes, reviewing their construction and theory on model equations which range from the simple advection equation to Hamilton–Jacobi equations. In the basic situation of first-order schemes, we will also make a comparison with more standard approximation techniques, namely finite difference schemes of upwind and centered types. The reader will find information on the construction of the schemes together with a detailed theoretical analysis, at least for the more popular choices of their elementary building blocks. We have included a number of practical algorithmic recipes, and we believe that after reading this book it should be possible to program and apply the methods described here from scratch.

The second goal of the book is to present some applications and show the Semi-Lagrangian approximations at work on a variety of nonlinear problems of applicative interest. We also aim at showing that this technique is well suited for the approximation of problems with nonsmooth solutions, like weak solutions in the viscosity sense in the case of Hamilton–Jacobi equations.

The book is intended to be accessible to a reader with a basic knowledge of linear PDEs and standard numerical methods. The analysis of Hamilton–Jacobi equations will, of course, require specific analytical tools, which have been included in the section devoted to viscosity solutions. The presentation is conceived to be reasonably self-contained, but at the same time it will collect in a unified framework results which are spread over the literature. Clearly, more information on the theory of viscosity solutions can be obtained looking at the specific references, and in particular at the books [Ba98, BCD97, FS93]. Some standard finite difference schemes for Hamilton–Jacobi equations can be found in [Se99, OF03], whereas other schemes based on control techniques are presented in [KuD01].

Although Hamilton–Jacobi equations and their applications are the object of an extensive literature, we will mostly limit our presentation to the Semi-Lagrangian approach and to its relationship with Dynamic Programming. A detailed overview of numerical methods for Hamilton–Jacobi equations goes beyond the scopes of this book, although we hope to develop this topic in greater detail in the future.

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