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Iterative Methods for Linear Systems: Theory and Applications. *By Maxim A. Olshanskii and Eugene E. Tyrtyshnikov.* SIAM, Philadelphia, 2014. $85.00, xvi+247 pp., softcover. ISBN 978-1-611973-45-7.

With so many good books on iterative methods published in the 1990s and early 2000s, e.g., [1, 9, 11, 12, 14], it was surprising to see a new book on this subject, and I wondered what new material it might contain. The answer is that there is not a lot new on the basic iterative methods—conjugate gradients, GMRES, BiCG, QMR, etc., are the same as they were 10 years ago—but this book contains new material on preconditioners, Toeplitz and circulant matrices, and multigrid and domain decomposition methods, with a very nice arrangement of topics to tie the ideas together. The theoretical analysis is rigorous and makes use of a wide range of diverse techniques. The book would be difficult for someone who has little prior knowledge of iterative methods, but for those who are familiar with the basics and wish to learn more about analysis and applications, it is an excellent resource.

Chapter 1 deals with Krylov subspace methods and presents rigorous mathematical theory of convergence as well as optimality of approximations. Convergence estimates for nonnormal matrices are presented in terms of pseudospectra and the numerical range. I had hoped to see some of the more recent estimates such as those derived by Beckermann et al. (including one of the authors of this book) [2, 3, 4] and some discussion of the implications of Crouzeix's theorem or conjecture [7, 8] for the convergence of GMRES. Still, the theory that is presented is stated clearly and proved elegantly. It also might have been interesting to see some mention of the recently popularized IDR method [13, 10]. Algorithms are presented in a form that is easy to understand but not always best for computation. (For example, the unmodified Gram–Schmidt version of GMRES is given, with no discussion of how to update the QR factorization of the Hessenberg matrix.) The authors state that their book should be considered complementary to other books on iterative methods, and this is one of the areas in which other books should be consulted.

Chapter 2 deals with Toeplitz matrices and preconditioners, a topic that typically receives less attention in books on iterative methods. Simple and optimal circulant preconditioners are defined and analyzed, and the relation with the fast Fourier transform is explained. The topic of multilevel Toeplitz and circulant matrices is also addressed and leads naturally into the following chapters on multigrid and domain decomposition methods for problems arising from partial differential equations.

Chapter 3 gives a very nice introduction to multigrid methods, used either as stand-alone solvers or as preconditioners for conjugate gradient or other iterative methods. Relevant properties of finite element approximations are clearly delineated and then used to provide a clear and succinct proof of convergence (at a rate that is independent of the mesh size; i.e., with error reduction factor $1/2$ at each step) for a two-grid method applied to the model problem of Poisson's equation. This is a chapter that I would recommend to students with or without a background in multigrid methods. I often recommend *A Multigrid Tutorial* by Briggs et al. [6], but while that book provides a very friendly, easy-to-read introduction to how multigrid methods work, the lack of mathematical analysis can lead to confusion. This book makes the mathematics very clear. I was especially interested in the proof (p. 113) that the 2-norm of the iteration matrix for the model problem is less than or equal to $1/2$, as proofs that I had seen previously dealt with the A-norm of the iteration matrix. After the section on the two-grid method there is a nice explanation of how two-grid theory can be used to establish convergence of the multigrid V-cycle or W-cycle.

Chapter 4 deals with space decomposition methods, such as alternating Schwarz methods, domain decomposition methods, and the additive multigrid method viewed as a space decomposition algorithm. There is discussion of the BPX (Bramble–Pasciak–

Xu) [5] preconditioner and also of hierarchical basis methods [15]. The unifying theme is decomposition of the problem into different parts (domains, frequencies, etc.), and this point of view yields a unifying analysis.

The final chapter in the book discusses applications and modifications of the general theory that are often required to develop algorithms that work in practice. Problems considered include singularly perturbed partial differential equations and certain problems in fluid mechanics. There is discussion of different multigrid smoothers that may be needed in order to obtain convergence rates that are independent of the mesh size. Examples are ILU smoothers and a number of others designed for specific types of problems.

Overall, I enjoyed this book very much. There are a number of typos and occasional changes in notation (e.g., sometimes the grid size is $1/N$ and sometimes it is $1/(N+1)$) and I wish the authors would avoid the expression $\mathrm{cond}(BA)$ for the ratio of the largest to smallest eigenvalue of the preconditioned matrix BA (or AB) in the conjugate gradient algorithm. This leads to confusion in Theorem 1.33 and the discussion afterwards, where the norm being used is not specified. Hopefully these small issues can be fixed if a second edition appears. This book demonstrates the blending that has taken place during the last 20 years or so of what were once considered somewhat separate and maybe even competing ideas—iterative methods, multigrid, and domain decomposition—into one cohesive whole. The mathematical theory is there, and there are helpful exercises at the end of each chapter to drive home the main points. It is a valuable new resource for the research community in iterative methods.

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