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Solving Transcendental Equations: The Chebyshev Polynomial Proxy and Other Numerical Rootfinders, Perturbation Series, and Oracles. *By John P. Boyd.* SIAM, Philadelphia, 2014. $99.00. xviii+460 pp., softcover. ISBN 978-1-611973-51-8.

I reviewed a prepublication draft of this book (hereinafter STE) for the SIAM editors and suggested significant changes before publication. I "pulled no punches" but did recommend publication once some issues were addressed. They were, of course, as were those of other reviewers.

STE as published is quite different from that earlier draft, and I am very happy to have this chance to review the finished product as well. STE is very much worth reading twice anyway, containing as it does many gems of the art and science of solving nonlinear equations. It concentrates on the scalar case—one equation in one unknown—but also has a valuable chapter on the next case, two equations in two unknowns. STE is written in a lively, informal style, and the author is an expert. The book's stated aim is "to teach the art of finding the [*sic*] root of a single algebraic equation or a pair of such equations." STE sticks to topics related to that goal, and I can say, at least, that I learned several things.

As you are no doubt aware, rootfinding is indeed more of an art than a science. The field is littered with impossibility results, and where algorithms are possible their complexity can make them "unaffordable," to use terminology that I only recently learned but which I like. Even very short equations, such as

$$z + v \csc v \, e^{-v \cot v} = 0,$$

can have ridiculously complicated sets of solutions. The above equation is to be solved for v, given z. The solutions are, surprisingly, expressible in a short form:

$$v_{k\ell} = \frac{1}{2i}(W_k(z) - W_\ell(z)),$$

where $W_k(z)$ is the kth branch of the Lambert W function. A proof of the above can be found in [1]. Evaluation of the Lambert W function itself requires rootfinding, of course, but its roots can be enumerated easily with a single index, while the roots of this equation seem to be best expressed with two. There are accumulation points of $v_{k\ell}$ at nonzero odd multiples of π, by the way. (This example is not from STE and would be a challenge for any numerical method. STE does do a very nice job of describing the Lambert W function, though.)

Any purported rootfinder has to be prepared, then, for remarkable complexity. It's not a surprise that to make progress we have to make restrictions on the class of problems tackled. Different restrictions lead to different prescriptions of what to do.

A common and important restriction is to look for all real roots on an interval. STE discusses in ample detail what I believe to be the best general-purpose method for finding all such roots, namely, the Chebyshev Proxy Rootfinder, or CPR. The idea of the method is, in principle, simple. Approximate your function by a truncated Chebyshev series. You may have to subdivide your interval for efficiency's sake. Find the real roots on each subinterval using a Chebyshev companion matrix. The idea of using Chebyshev approximations and companion matrices to find real roots is apparently due to I. J. Good [2], but Boyd's contribution—implementation and refinement by using subdivision—seems to have been decisive. In [3] Boyd gave it the name CPR. This method has been implemented in the Chebfun package (www.chebfun.org and [4]), so it's not just theory. The ability of Chebfun to find real roots is, frankly, amazing. We give an example on pp. 150–154 of our book [5] where 999 roots of a discontinuous function are found to an accuracy of better than $5\epsilon_M$ using Chebfun.

The method thst Chebfun uses, which is essentially CPR, is explained in STE and also in [3].

I point out that STE contains that paper, [3], in its entirety, but also contains considerably more. The material from [3] only takes up the first three chapters of STE, out of twenty.

The other chapters cover Fourier interpolation and companion matrices, complex zeros and contour integral methods of great recent interest, continuation methods, bifurcation theory, Newton-like methods (of course!), four chapters on solution by radicals or trigonometric formulae (which I think are very valuable chapters), series methods, and perturbation methods. The bivariate chapter was and is especially interesting to me (my student Azar Shakoori wrote Maple's bivariate polynomial solver); I learned a lot from that chapter that I wish I had known then. There's still a lot left to learn.

I wanted to choose an example from the many in STE to show the flavor of the book. I found that I couldn't decide which one I liked best! Expansion of the complex zeros of the error function (pp. 270–272)? A singular perturbation example such as the Laplacian eigenvalue in an annulus (p. 296)? A Never-Failing Newton Initialization (NFNI) for an equation containing an elliptic integral (p. 140)? Indeed I quite liked the notion of an NFNI, which was new to me. I find the example-based format of STE quite conducive to readability and therefore utility; choosing just one example wouldn't, after all, give the flavor of this smörgåsbord.

I recommend this book, especially to people who are trying to find and implement algorithms that work well for a particular class of problem. I also recommend it (or [3]) to people who want to know more details of the ideas underlying how Chebfun finds roots.

Rootfinding is an active area. STE doesn't cover the whole of it, and in particular is not concerned with large systems of equations. (For large polynomial systems, see [6], [7]; for high-degree polynomials see the MPSolve project [8] and [9]; for work in the Bernstein–Bézier basis, see, e.g., [10]; and that's only the tip of the iceberg.) But I believe STE to be a valuable addition to the literature. I certainly found it fun to read.

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