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**Spline Functions: Computational Methods.** *By Larry L. Schumaker.* SIAM, Philadelphia, PA, 2015. $83.00. xii+412 pp., hardcover. ISBN 978-1-611973-89-1.

The book is a long-awaited sequel to two previous books on splines, [2] and [1], by the same author. The new book is a product of over fifty years of research, teaching, and collaboration with numerous scientists within and outside the field of splines. The complete bibliography comprised of over 100 pages was too long for the hard copy and has been put online; see [3]. A very valuable part of the book is a MATLAB package, *SplinePak*, which is freely available both on [3] and on Larry L. Schumaker's website [4]. For the impatient reader, I would recommend downloading the package and going straight to the examples in the book, which, admittedly, is exactly what I did. Within minutes, I had a beautiful picture of a minimal energy quadratic smooth spherical spline interpolating $f(x, y, z) = x^4 + 1.1y^4 + 1.3z^4$ at 42 vertices of a triangulation of the unit sphere; see Figure 1. Each numerical example in the text describes a problem and has a reference to the code that provides a solution. The only drawback is that the MATLAB functions in *SplinePak* are currently p-files, and thus cannot be modified. The script files are the usual m-files. A complete list of the scripts and the functions in the package is included at the end of the book.

For the more patient reader, I recommend downloading the package and getting hold of both of Larry Schumaker's previous books on splines, *Spline Functions: Basic Theory* [2] and *Spline Functions on Triangulations* [1] (coauthored with M. J. Lai). This takes us to the next truly valuable feature of the book: it is equally useful to the reader looking for algorithms to solve practical problems and to the reader interested in rigorous foundations for such algorithms. While the book itself includes few proofs, it contains rigorous statements of all theorems used, and full references to their proofs elsewhere. Most of the referenced proofs can be found in [2] and [1]. I will demonstrate this approach by summarizing the content of one section—section 5.2—titled "The $C^1$ Powell–Sabin Interpolant." It begins by introducing the so-called Powell–Sabin refinement of a triangulation. The following theorem states interpolating conditions sufficient to define a unique interpolating $C^1$ quadratic spline on the Powell–Sabin refinement. Immediately after the theorem we find a reference to section 6.3 of [1], where the proof can be found. Next, we discover all the necessary formulae to define the coefficients of the interpolating spline, followed by a reference to a code in *SplinePak* that returns a vector of the coefficients of the spline along with a figure of the spline surface. Two numerical examples that are discussed next include the maximum and root mean square errors and the convergence rate analysis. The two examples treat an interpolation problem of Franke's function on a triangulation with 36 vertices, and on several refinements of type-I triangulation. The section concludes with the statement of a rigorous error bound in Sobolev norm followed by the general idea of the proof and a reference in [1], where the proof can be found.

The material covered by the book is broad. Piecewise polynomials (splines) in one or two variables can be used to solve approximation, interpolation, data fitting, numerical quadrature, and ODE and PDE problems, and this is exactly what the book shows the reader how to do. Chapter 1 deals with univariate splines, their evaluation, and their use in interpolation, approximation, and solving two-point boundary-value problems. Chapter 2 treats similar aspects of bivariate tensor-product splines. Theoretical foundations are covered in [2]. In Chapter 3 we learn how to deal with triangulations computationally. Chapter 4 is essential for understanding the rest of the book. It covers foundations of the Bernstein–Bézier representation of bivariate polynomials and splines. The first application of bivariate splines—Hermite interpolation—is demonstrated in Chapter 5, followed by scattered data fitting in Chapters 6, 7, and 8. Elliptic PDEs of orders two and four are the focus of Chapter 9, where the Ritz–Galerkin method is implemented with bi-

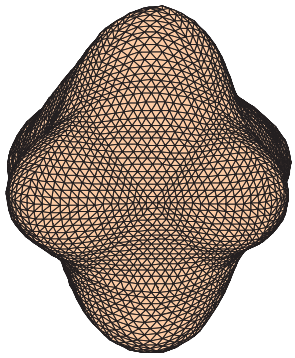

**Fig. 1** *A minimal energy spherical spline.*

variate macroelements. Chapters 9 and 10 cover spherical splines and their applications. Theoretical foundations for Chapters 3–5, 10, and 11 can be found in [1]. The bibliography included at the end of the book contains referenced books only, while the online one contains research papers and other resources.

Yet another aspect of this book that makes it very attractive is the fact that the research content is fully up to date. For example, the classical subject of computing triangulations includes recent developments on triangulations with hanging vertices. Solving a classical PDE problem using the finite element method bypasses the reference triangle by means of the Bernstein–Bézier representation and uses finite elements of higher degree and smoothness. Each chapter ends with remarks and historical notes, where all original sources are cited and additional research articles are referenced.

The book is an invaluable resource for many different types of readers, including researchers in the field of numerical analysis, applied mathematicians, computer scientists and engineers, and graduate students, as well as computational specialists from other sciences who are willing to apply splines to their fields.

## REFERENCES


[1] M.-J. Lai and L. L. Schumaker, *Spline Functions on Triangulations*, Cambridge University Press, Cambridge, UK, 2007.

[2] L. L. Schumaker, *Spline Functions: Basic Theory*, Wiley Interscience, New York, 1981.

[3] `www.siam.org/books/ot142`.

[4] `www.math.vanderbilt.edu/ schumake/`.



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