Preface

Spline functions have been studied intensively for over 50 years, and there are now many thousands of papers on the subject. The reason for this great interest is that splines have many desirable features that make them ideal tools for a wide variety of application areas where it is required to approximate an unknown function computationally. Thus, they have found applications in Approximation Theory, Numerical Analysis, Engineering, and a variety of other business, medical, and scientific areas. What makes splines especially attractive is the fact that

- they are easy to work with computationally, and in particular, there are stable and efficient algorithms for evaluating them, their derivatives, and their integrals;
- there are very convenient representations which provide a strong connection between the shape of a spline and its associated coefficients;
- splines are capable of approximating smooth functions well, and the exact relationship between the smoothness of a function and its order of approximation can be quantified.

The theoretical aspects of splines have been well documented in the literature; see the list of books at the end of this book and the online reference list. Some of them discuss applications, and a few even describe computational methods. The purpose of this book is to give a detailed treatment of some of the most important computational aspects of splines. It can be considered as a complement to my two earlier books on splines, neither of which dealt with applications or computations. The first, *Spline Functions: Basic Theory*, was first published in 1978 by Wiley Interscience, and deals only with univariate and tensor-product splines. A third edition with supplementary material and references was published in 2007 by Cambridge University Press. The second book, *Spline Functions on Triangulations*, was written with M.-J. Lai, and deals with bivariate splines defined on triangulations and quadrangulations, spherical splines defined on the sphere, and trivariate splines defined on tetrahedral partitions in $\mathbb{R}^3$. It was also published in 2007 by Cambridge University Press.

Splines can be used to solve several of the most basic problems in numerical analysis. The topics discussed here include

- interpolation in one or several variables,
- data fitting in one or several variables,
- numerical quadrature, i.e., the computation of derivatives and integrals based only on discrete data,

- the numerical solution of problems involving ordinary or partial differential equations.

The goal of this book is to not only describe a variety of methods for attacking these problems using splines, but to also explain why the methods work, and to give actual numerical examples. The aim is to describe each method in enough detail to allow the reader to program it himself. However, to enhance the learning process, and to encourage more experimentation with splines, I have prepared a MATLAB® package *SplinePak* which includes implementations of all of the methods discussed in the book. The package includes an extensive collection of p-files containing MATLAB functions for carrying out the various algorithms in the book, a set of scripts provided as m-files for carrying out specific tasks, and finally a number of data files used in the numerous examples in the book. The package is licensed by *Nashboro Press*; see `www.splinepak.com`. It should be emphasized that the code in the package is not production code, and has not been designed to achieve maximal efficiency. Indeed, for large scale computation, the user should consider coding the algorithms in another language.

The book is primarily about bivariate polynomial splines on triangulations, but for the sake of completeness, and to provide numerical tools for comparison purposes, I have included chapters on univariate splines, tensor-product splines, and splines on spherical triangulations. The material is organized as follows.

Chapter 1 deals with univariate splines. First, we discuss algorithms for evaluating splines and their derivatives, for computing integrals of splines, and for plotting splines. Then we discuss several methods for solving the basic Lagrange and Hermite interpolation problems in one variable. In addition, quasi-interpolation, least-squares, penalized least-squares, and the use of splines for solving two-point boundary-value problems are treated.

Chapter 2 is devoted to the use of tensor-product splines to solve interpolation and approximation problems in the bivariate setting. First, it is shown how univariate algorithms can be used to evaluate tensor-product splines, and to compute their derivatives and integrals. Then we study interpolation, quasi-interpolation, least-squares, and penalized least-squares methods based on tensor-product splines.

Chapter 3 explains how to deal with triangulations computationally, and is a critical preparation for the later chapters. Here we explain how to construct, store, and manipulate triangulations. Refinement and edge swapping algorithms are also included.

Chapter 4 describes in detail how to store, evaluate, and render splines on triangulations. The algorithms described here are essential tools for the remainder of the book. The key underlying concept is the Bernstein–Bézier representation of splines. Here we explain how to compute with the space $\mathcal{S}_d^0(\Delta)$ of C^0 splines, and with arbitrary subspaces of smoother splines. Several special macro-element spaces to be used extensively later are discussed in detail.

Chapter 5 is the first chapter where real applications are addressed. Here we show how certain macro-element spaces can be used to solve interpolation problems in the bivariate setting. The focus is on Hermite interpolation at the vertices of a triangulation. I give numerical examples for each of the methods, and include the results of some experiments comparing the various interpolation methods.

Chapter 6 treats the problem of scattered data fitting where no derivative information is available. Both local and global methods are discussed, beginning with minimal energy methods which may be the most useful for the typical interpolation problem. This is followed by a discussion of some special local schemes which are quite useful for larger problems. We then treat two-stage local methods based on the Hermite interpolation methods of the previous chapter. The idea of these methods is to estimate the needed derivative information directly from the scattered data. For comparison purposes, interpolation with radial basis functions is also discussed. In addition to numerous numerical examples, the results of some experiments comparing the various methods are included.

Chapter 7 is about the fitting of data as opposed to interpolating it. This approach is generally preferable for large amounts of data, or when we have noisy data. Both least-squares and penalized least-squares methods are treated, along with many numerical examples and a comparison of the various methods.

Chapter 8 is concerned with methods for interpolating and fitting bivariate data with shape control. Here we are looking for splines which are nonnegative, monotone, or convex. Problems of this type are of considerable importance in applications.

Chapter 9 is devoted to the numerical solution of boundary-value problems using bivariate splines with the Ritz–Galerkin (finite-element) method. This is a well-established technique in the Engineering community. The implementations there typically make use of isoparametric mappings with a heavy emphasis on C^0 piecewise polynomial spaces. Here we show how to work with spline spaces with higher order smoothness directly without using such mappings. The key is to use the Bernstein–Bézier representation. The focus is on elliptic problems of orders two and four.

Chapter 10 discusses spaces of splines defined on the sphere which are made up of piecewise spherical harmonics defined on spherical triangulations. In particular, it is shown how the methods for storing and manipulating bivariate splines can be easily extended to spherical splines.

Chapter 11 shows how the various interpolation and data fitting methods discussed in Chapters 6–7 can be easily adapted to work on the sphere. In particular, we give algorithms for scattered data interpolation with either Lagrange or Hermite data, and for least-squares and penalized-least squares fitting. We also describe a method for solving partial differential equations on the sphere.

Each chapter of the book includes a section with remarks, where the aim is to provide interesting and useful tangential information without interrupting the flow of the book. Moreover, each chapter concludes with a section with historical notes designed to give the reader some insight into how the material of the chapter developed. Because of the huge literature, it was impossible to go into much detail.

A bibliography listing books on splines is included at the end of this book. Unfortunately, it was not practical to include a list of papers on splines related to the material in this book. It would have taken well over 100 pages. For the full list of references for the book, see `www.siam.org/books/ot142`. There is also a link there to an even more extensive bibliography of papers related to splines, maintained by Carl de Boor and the author. That site even includes files with the references in TEX form.

Since we have different reference lists for books and papers, we need a mechanism for distinguishing between the two in citations. We use double brackets for books, and single brackets for papers. Thus, [[LaiS07]] refers to a book whose details can be found in the `Bibliography`, but [AlfNS96] refers to a paper which is not listed here, but rather in the online reference set. Each reference is assigned a code based on the first three letters of the first author's name, along with the first letter of each coauthor. The code also includes the year of publication. In some cases we have used more than three letters for the first author in order to distinguish authors.

For convenience, after the `Bibliography` we have included an index of the scripts referred to in the examples given in the book. This is followed by an index of the functions used in these scripts. The MATLAB m-code of the scripts and the p-code for the functions called by them (along with some other supporting functions) are all included in the package *SplinePak*.

The book concludes with a detailed subject index. This list includes page numbers for all the words defined in the book as identified by this **special font**. Various other important concepts are also indexed. We do not attempt to list every page on which a given word appears.

To fit this book into space and time constraints, there are many topics that have not been discussed here. In particular, we have opted not to discuss computing with trivariate splines. There are also many specific applications of splines in the literature that have not been touched on here.

In many ways, the preparation of this book has stretched over the entire 50 years that the author has been working on splines. At some point, almost all of the algorithms described here were originally coded in Fortran. The material of the book has been used in many of my courses over the years, starting in 1966–1968 when I was a postdoc at the Mathematics Research Center in Madison. It was further developed for spline courses I taught later in Austin, Berlin, Sao Paulo, College Station, Munich, and for the last 25 years in Nashville.

Over the years, I have benefited from discussions of this material with many of my colleagues. I make no attempt to list them all here, but I would like to especially thank Peter Alfeld, Carl de Boor, Oleg Davydov, Manfred von Golitschek, Ming-jun Lai, and Tom Lyche. I would also like to thank my graduate students over the years for their interest and feedback. I am particularly grateful to my students Lujun Wang and Wenjia Zhang, who read the entire manuscript and checked all of the numerical examples.