

Contents

Preface	x ⁱ
Notation and Abbreviations	xv
I Riemann–Hilbert Problems	1
1 Classical Applications of Riemann–Hilbert Problems	3
1.1 Error function: From integral representation to Riemann–Hilbert problem	5
1.2 Elliptic integrals	7
1.3 Airy function: From differential equation to Riemann–Hilbert problem	9
1.4 Monodromy	11
1.5 Jacobi operators and orthogonal polynomials	13
1.6 Spectral analysis of Schrödinger operators	16
2 Riemann–Hilbert Problems	23
2.1 Precise statement of a Riemann–Hilbert problem	23
2.2 Hölder theory of Cauchy integrals	25
2.3 The solution of scalar Riemann–Hilbert problems	34
2.4 The solution of some matrix Riemann–Hilbert problems	40
2.5 Hardy spaces and Cauchy integrals	44
2.6 Sobolev spaces	59
2.7 Singular integral equations	63
2.8 Additional considerations	78
3 Inverse Scattering and Nonlinear Steepest Descent	87
3.1 The inverse scattering transform	88
3.2 Nonlinear steepest descent	97
II Numerical Solution of Riemann–Hilbert Problems	107
4 Approximating Functions	109
4.1 The discrete Fourier transform	109
4.2 Chebyshev series	115
4.3 Mapped series	118
4.4 Vanishing bases	120

5	Numerical Computation of Cauchy Transforms	125
5.1	Convergence of approximation of Cauchy transforms	126
5.2	The unit circle	128
5.3	Case study: Computing the error function	130
5.4	The unit interval and square root singularities	131
5.5	Case study: Computing elliptic integrals	135
5.6	Smooth functions on the unit interval	136
5.7	Approximation of Cauchy transforms near endpoint singularities . .	144
6	The Numerical Solution of Riemann–Hilbert Problems	155
6.1	Projection methods	156
6.2	Collocation method for RH problems	160
6.3	Case study: Airy equation	167
6.4	Case study: Monodromy of an ODE with three singular points . .	168
7	Uniform Approximation Theory for Riemann–Hilbert Problems	173
7.1	A numerical Riemann–Hilbert framework	175
7.2	Solving an RH problem on disjoint contours	177
7.3	Uniform approximation	182
7.4	A collocation method realization	187
III	The Computation of Nonlinear Special Functions and Solutions of Nonlinear PDEs	191
8	The Korteweg–de Vries and Modified Korteweg–de Vries Equations	193
8.1	The modified Korteweg–de Vries equation	202
8.2	The Korteweg–de Vries equation	209
8.3	Uniform approximation	227
9	The Focusing and Defocusing Nonlinear Schrödinger Equations	231
9.1	Integrability and Riemann–Hilbert problems	232
9.2	Numerical direct scattering	234
9.3	Numerical inverse scattering	237
9.4	Extension to homogeneous Robin boundary conditions on the half-line	241
9.5	Singular solutions	246
9.6	Uniform approximation	247
10	The Painlevé II Transcendents	253
10.1	Positive x , $s_2 = 0$, and $0 \leq 1 - s_1 s_3 \leq 1$	256
10.2	Negative x , $s_2 = 0$, and $1 - s_1 s_3 > 0$	258
10.3	Negative x , $s_2 = 0$, and $s_1 s_3 = 1$	263
10.4	Numerical results	265
11	The Finite-Genus Solutions of the Korteweg–de Vries Equation	269
11.1	Riemann surfaces	270
11.2	The finite-genus solutions of the KdV equation	274
11.3	From a Riemann surface of genus g to the cut plane	278
11.4	Regularization	279
11.5	A Riemann–Hilbert problem with smooth solutions	284

11.6	Numerical computation	289
11.7	Analysis of the deformed and regularized RH problem	297
11.8	Uniform approximation	300
12	The Dressing Method and Nonlinear Superposition	303
12.1	A numerical dressing method for the KdV equation	304
12.2	A numerical dressing method for the defocusing NLS equation . .	315
IV	Appendices	321
A	Function Spaces and Functional Analysis	323
A.1	Banach spaces	323
A.2	Linear operators	330
A.3	Matrix-valued functions	332
B	Fourier and Chebyshev Series	333
B.1	Fourier series	333
B.2	Chebyshev series	337
C	Complex Analysis	345
C.1	Inferred analyticity	345
D	Rational Approximation	347
D.1	Bounded contours	347
D.2	Lipschitz graphs	348
E	Additional KdV Results	357
E.1	Comparison with existing numerical methods	357
E.2	The KdV g -function	359
	Bibliography	363
	Index	371