Preface

Matrix theory is one of the most fundamental tools of mathematics and science, and a number of classical books on matrix analysis have been written to explore this theory. As a higher order generalization of a matrix, the concept of tensors or hypermatrices has been introduced and studied due to multi-indexed data sets from wide applications in scientific and engineering communities. With more subscripts, compared to matrices, tensors possess their own geometric and algebraic structures which might be lost if we reshape or unfold them into matrices. One of their intrinsic features that heavily relies on the tensor structures is the concept of tensor eigenvalues, which turns out to be much more complex than that of the matrix case. Thus, tensors must then be treated as data objects in their own right, and theory on this new type of objects is required, while matrix analysis is still of importance but less so.

Our purpose in writing this book is to present a relatively systematic treatment of tensors with great emphasis on spectral properties (properties of eigenvalues) and four special types of tensors including nonnegative tensors, positive semidefinite tensors, completely positive tensors, and copositive tensors. As a very essential and important theoretical application of tensor eigenvalues and eigenvectors, spectral hypergraph theory, has attracted much attention and evolved rapidly over the last few years. A very careful treatment of spectral hypergraph theory is therefore presented in this book. Besides spectral hypergraph theory, wide applications of tensors have been found in automatic control, higher order Markov chains, polynomial optimization, magnetic resonance imaging, quantum entanglement problems, multilinear systems, tensor complementarity problems, tensor eigenvalue complementarity problems, hypergraph partition, etc. Some of them will be briefly mentioned in relevant places throughout the book for reference.

This book is divided into six chapters. The first two chapters are devoted to some basic knowledge about tensors. Chapters 3, 5, and 6 target four special types of tensors which have found applications in a variety of disciplines and areas. Chapter 4 is about spectral hypergraph theory via tensors, heavily based upon a great deal of analysis on adjacency tensors and (signless) Laplacian tensors of hypergraphs. This builds a bridge between tensors and hypergraphs and possibly sheds a light on the tensor-level thinking of further and deeper research on hypergraphs. Each chapter of the book starts with a brief introduction to the contents of the chapter right before the opening section, and ends up with some notes and exercises as the last two sections. Particularly, at several places, some unsolved research problems have also been pointed out to encourage readers toward further research. Although this book focuses on theories, algorithms are studied in each chapter after the introduction chapter.

The existing research work on this new theory of tensors is scattered among various papers, which is inconvenient for researchers in or out of this area, especially for graduate students who wish to study this new theory. A comprehensive reference book was therefore needed, which came to be the very reason that we decided to write this book. This book can be used as a textbook for graduate courses and also be useful as a reference for researchers in tensors and numerical multilinear algebra.

We assume that readers of this book have a sound knowledge of matrix theory and linear algebra, and almost no advanced prerequisites are needed, albeit some tools from algebraic geometry might be involved such as the concept of resultants in the analysis of characteristic polynomials and determinants of tensors, and moment theory involved in algorithms for polynomial programming. We would advise readers in their first reading of this book to safely skip these parts, at least temporarily, without causing any gap in understanding. Each of the involved sections and subsections, including the whole Section 3.4, and Subsections 2.1.4, 2.1.5, 2.4.1, 3.3.3–3.3.5, 5.7.4, and 5.7.5, is marked with an asterisk for readers' convenience. The tensor decomposition, which is definitely an essential ingredient of tensor analysis and has a sound theory that can be found in some survey papers or books, will no longer be elaborated again in this book. To clearly point out our emphasis, we named our book *Tensor Analysis: Spectral Theory and Special Tensors*. Comments are very welcome so that we may further improve this book.

In a certain sense, this book aims to realize the dreams of Arthur Cayley (1821–1895), Israel Gelfand (1913–2009), and Gene Golub (1932–2007) to extend matrix theory to hypermatrices. Arthur Cayley was one of the founders of matrix theory, best known for the Cayley-Hamilton theorem in matrix theory. Less known about him is that he proposed the concept of hyperdeterminants and planned to pursue further in this direction [68]. Israel Gelfand was regarded as one of the top mathematicians in the last century. In the preface of his book *Discriminants, Resultants, and Multidimensional Determinants* with Kapranov and Zelevinsky [161], he stated clearly Cayley's dream and the relation with their book. Gene Golub was the world leader of numerical linear algebra. A comprehensive exposition of his remarkable contributions can be found in his classical book *Matrix Computations* [170]. As a preeminent numerical analyst, Gene started to promote the development of tensor decomposition for multiway data to address the urgent need arising from modern real-life problems in the big data era. Profoundly influenced by Gene's work on tensor decomposition and Israel Gelfand's aforementioned book, Liqun Qi introduced eigenvalues of tensors in 2005 [381]. After reading this work, Gene visited Qi in Hong Kong in September 2005 and on November 1, 2007, and invited Qi to conferences and workshops at Stanford and in Europe. Unfortunately, Gene passed away while at Stanford on November 16, 2007.

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We have benefited from our recent discussions with Daniel Braun, Epifanio Virga, and Kristjan Kannike. We learned from Kristjan Kannike that copositive tensors play an important role in study of quantum field theories [245]. We clarify now that a special positive semidefinite tensor class plays a significant role in the study of classicality of spin states [34, 167], and some third-order three-dimensional symmetric traceless tensors are important in the study of liquid crystal [158, 449]. We truly expect that the further study of this subject will be much more application driven.

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