may choose $\mathcal{N}^{(0)}$ such that $\mathcal{N}^{(0)}$-leading becomes $\prec$-leading for a particular term order; then the associated border basis may become the Groebner basis of P for that term order. Without a term order, one may rather wish to avoid high degrees in the normal set.

Example 10.6: Consider a system P in 3 variables of a dense cubic polynomial p_1 and two dense quadratic polynomials p_2, p_3. We have $m = 12$; thus we must choose a closed set $\mathcal{N}$ of 12 monomials in $\mathbb{N}_0^3$ such that one monomial $x^{j^{(\nu)}}$ of each $p_{0\nu}$ is outside $\mathcal{N}$ (and the $x^{j^{(\nu)}}$ are disjoint). A natural choice is $\mathcal{N} = \{x_1^{j_1} x_2^{j_2} x_3^{j_3}$, with $j_1, j_2 \le 1,\ j_3 \le 2\}$, and $x^{j^{(1)}} = x_1^2,\ x^{j^{(2)}} = x_2^2,\ x^{j^{(3)}} = x_3^3$; cf. Figure 10.2-1.

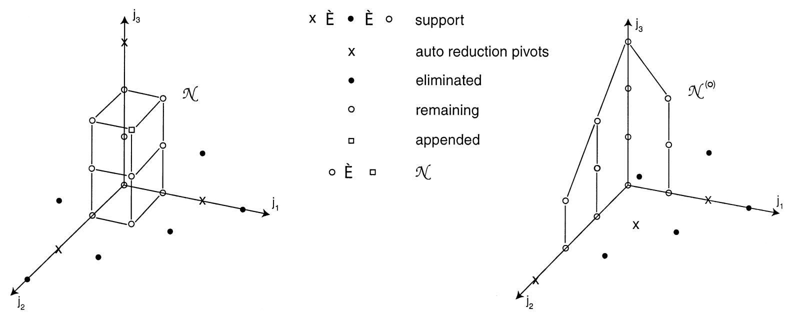

Figure 10.2-1 **Figure 10.2-2**

In the autoreduction phase, the x_2^2-term of p_2 and the x_1^2-term of p_3 are removed and the remaining 3rd degree monomials of p_1 outside $\mathcal{N}$ are removed by reduction with p_{02} and p_{03}; now the $p_{0\nu}$ and $\mathcal{N}$ satisfy (10.4). The union of the supports of the tails has only 11 elements; but we had attached the further monomial $x_1 x_2 x_3^2$ to $\mathcal{N}$ from the beginning. Thus $\mathcal{N}$ is a valid candidate; it will turn out that it is feasible and supports a border basis.

The strong symmetry of this normal set prevents it from supporting a Groebner basis for any term order. In an attempt to comply with, say, `tdeg(x1,x2,x3)` we can choose the two $\prec$-highest monomials in p_2, p_3 for the autoreduction which makes x_1^2 and $x_1 x_2$ the leading monomials of p_{02} and p_{03}; this permits the removal of their multiples from p_1 and makes x_2^3 the $\prec$-leading monomial of p_{01}; cf. Figure 10.2-2. Now, the union of the supports of the tails has precisely 12 elements, which determine our candidate normal set $\mathcal{N}^{(0)}$. In the construction process of the border basis, it will turn out that $\mathcal{N}^{(0)}$ is not feasible but has to be modified in one position.

For both autoreduced systems P_0, it is obvious that $\langle P \rangle = \langle P_0 \rangle$: In both cases, p_2, p_3 are scalar linear combinations of p_{02}, p_{03}, and $p_1 = p_{01} + q_2\, p_{02} + q_3\, p_{03}$. Thus, each of the 12 zeros of P_0 is a zero of P which cannot have more than 12 zeros. □

Obviously, one should begin the autoreduction with the polynomial(s) of lowest degree because they represent the strongest restriction on the choice of the normal set. Without a term order, the candidate normal set can generally be enclosed within an s-dimensional rectangle determined by monomials of highest degree from the p_ν. If $\mathrm{BKK}(P) < \prod_\nu d_\nu$, the autoreduction has to be watched more carefully so that the right monomials are removed.

Concerning the requirement $\langle P \rangle = \langle P_0 \rangle$, we have