Understanding and Implementing the Finite Element Method. *By Mark S. Gockenbach.* SIAM, Philadelphia, 2006. $87.00. xvi+363 pp., softcover. ISBN 978-0-898716-14-6.

This book will make an excellent text for an undergraduate applied mathematics class on the numerical solution of partial differential equations by the finite element method. It contains a balanced blend of numerical methods, theory, and implementation considerations. I was particularly pleased to see the treatment of modern techniques, like preconditioned conjugate gradients, multigrid, and adaptive grid refinement, as mainstream topics. This is not a text for a class on programming for scientific computing; it is intended for a class on the finite element method. The accompanying MATLAB programs (available by download from the web) allow students to experiment with aspects of the finite element method without spending hours programming in a language like Fortran or C. A more theoretically oriented numerical analysis class will find the lack of proofs for most theorems disturbing, although references where the proofs can be found are always given. On the other extreme, an engineering class may find the mathematics a little deep, but most of it can be skipped or deemphasized while still learning sufficient basics to understand the practical implementation details.

The book is organized in four parts: the finite element method, implementation, solution of linear systems, and adaptive grid refinement. It focuses on the model 2D linear elliptic boundary value problem

$$\begin{aligned} -\nabla \cdot (\kappa \nabla u) &= f \text{ in } \Omega, \\ u &= g \text{ on } \Gamma_1, \\ \kappa \frac{\partial u}{\partial n} &= h \text{ on } \Gamma_2, \end{aligned}$$

where Ω is a bounded region in the plane with boundary $\Gamma_1 \cup \Gamma_2$, and κ, f, g, and h are scalar functions of x and y. Other problems are mentioned, and sometimes addressed, but the book rightly focuses on the model problem in order to be concrete. Also the finite element method is limited to the Galerkin finite element method with C^0 piecewise polynomials, primarily with triangle meshes.

In Part I, the finite element method is developed. This is a nice presentation beginning with the weak form of the boundary value problem and the relevant Hilbert spaces. The development proceeds through the Galerkin method to the definition of piecewise polynomial spaces, including isoparametric elements. The final chapter in this part analyzes the convergence of the finite element method. The level of theory in this part seems appropriate for an applied mathematics class. The necessary math to provide motivation, develop the methods, and derive the equations is there, but lengthy proofs of theorems are omitted. The exercises at the end of each chapter in this part are mathematical, usually beginning with the word "derive," "show," or "prove."

Part II addresses the practical implementation of the finite element method. The algorithms and mesh data structure are presented in such a way that they could be implemented in a straightforward manner in any procedural language like Fortran or C. The focus is on clarity, not efficiency. The actual implementation of the algorithms is given in MATLAB, and they are available for download. The implementation is limited to the 2D model problem and triangular elements, although the last chapter in this part considers some other problems. Solution of the resulting linear system is postponed until the next part (the MATLAB solver is used at this point), but all other details of the implementation are addressed, including numerical quadrature, high order elements, and isoparametric elements for curved boundaries. The exercises in this part are a mixture of mathematical problems, operation/memory counts, using the given MATLAB programs, and writing new MATLAB programs.

Part III covers the solution of the linear system of equations that is generated by the finite element method. Direct methods are presented to contrast them to iterative methods. Stationary iterative methods are presented so that they can be used as preconditioners and smoothers, and also to contrast the rate of convergence. The preconditioned conjugate gradient and multigrid methods are presented as the preferred solution methods. A limited number of preconditioners are discussed. The hierarchical basis preconditioner is featured, and brief mention is given to diagonal scaling, SSOR, incomplete Cholesky, and fast Poisson solver preconditioners. A straightforward geometric multigrid method is derived using interpolation for the prolongation operator and its transpose for restriction. Much more could have been said about different types of multigrid methods (including algebraic multigrid) and other preconditioners for conjugate gradients, but at least the student gets a taste of modern solution methods for linear systems. Examples are given to illustrate the rate of convergence of these methods. Exercises in this part include both mathematical problems and the use of given MATLAB routines.

Part IV addresses adaptive methods. The focus is on the development of an adaptive mesh refinement algorithm using newest node bisection of triangles. This includes element-by-element error estimators, a strategy for choosing which triangles to refine, and the local refinement of the mesh. Three error estimators are derived. Examples are given to illustrate the advantage of adaptive grids. Exercises in this part are a mixture of mathematical problems, using the given MATLAB programs, and writing new MATLAB programs.

There are many topics not covered in this book, such as parallel implementation, domain decomposition, and 3D problems, and some of the topics covered could have more breadth, such as the presentation of more preconditioners and other multigrid methods. But to do so would distract from the main purpose of the book, which is to present the foundations of the finite element method and how to implement it in a succinct and concrete manner. Overall, I think this book will make an excellent text for its intended audience.

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Computing with hp-Adaptive Finite Elements. Volume I: One and Two Dimensional Elliptic and Maxwell Problems. *By Leszek Demkowicz.* Chapman and Hall/CRC, Boca Raton, FL, 2007. $89.95. xxvi+398 pp., hardcover. ISBN 978-1-58488-671-6.

The finite element method has become part of the standard computational toolbox for solving partial differential equations. However, even with the advent of faster processors, larger memory, and parallel computing, standard off-the-shelf methods may not work. Effective use of computational resources is still essential in solving difficult problems in two and three dimensions. Equally important, reliable results cannot be obtained without intelligent use of these same resources. Adaptive finite element methods offer one possible solution to these two problems while requiring minimal user intervention. There are few texts on adaptive finite element methods. This volume makes a fundamental contribution to the field of *hp*-adaptive finite element methods. It is not a survey or summary of *hp*-adaptive methods but a presentation of the author's approach. Nevertheless, I consider it the best current book in the area. It provides a greater level of detail, especially regarding the algorithms, than one typically finds in finite element texts (and in some spots I would have liked even more detail). The author also develops the theoretical underpinnings that enhance the reliability of his approach. I highly recommend this book for anyone interested in learning about adaptive finite element methods.

Adaptive finite element codes have been under development for almost three decades. Adaptive methods seek to enrich (refine) the discretization where certain features of the solution need to be enhanced and to coarsen it in regions where little of interest is occurring. Two adaptive strategies have become commonplace. They are often referred to as h- and p-refinement. In h-refinement elements are either added or removed. With p-refinement the order of approximation on an element is increased or decreased. p-refinement offers more rapid convergence but requires higher solution regularity. h-refinement is more appropriate in regions where the solution is less smooth. In two and three dimensions anisotropic h- and p-refinements that align with features of interest are advantageous. *hp*-adaptive methods seek to combine the strengths of both approaches. Work by Babuška and coworkers [5, 6, 7, 8, 9] demonstrated that for a certain class of problems exponential